gAGE/UPC research group of Astronomy and Geomatics
BarcelonaTECH, Spain

Lecture 13

Ambiguity Resolution Techniques

Professors: Dr. J. Sanz Subirana, Dr. J.M. Juan Zornoza
and Dr. A. Rovira-García

Contents

Ambiguity resolution Techniques

1. Resolving ambiguities one at a time
   - Single-frequency measurements
   - Dual-frequency measurements
   - Three-frequency measurements
2. Resolving ambiguities as a set: Search techniques
   - Least-Squares Ambiguity Search Technique.
   - LAMBDA Method.

Ambiguity resolution Techniques

As a driven problem to study the ambiguity fixing, we will consider the problem of differential positioning in DD for **short baselines** (e.g. $< 10\ \mathrm{km}$). In general we will consider that we have Code and Carrier measurements in different frequencies ($q=1,2\ldots$), i.e. $P_1, P_2, L_1, L_2\ldots$

$$P_{q,ru}^{jk} = \rho_{ru}^{jk} + T_{ru}^{jk} + I_{q,ru}^{jk} + \nu_{P_q,ru}^{jk}\ ; \quad q=1,2\ldots$$

$$L_{q,ru}^{jk} = \rho_{ru}^{jk} + T_{ru}^{j} - I_{q,ru}^{jk} + \lambda_q\, \omega_{ru}^{jk} + \lambda\, N_{q,ru}^{jk} + \nu_{L_q,ru}^{jk}$$

Short baseline

$$T_{ru}^{j} \approx 0$$
$$I_{q,ru}^{jk} \approx 0$$
$$\omega_{ru}^{jk} \approx 0$$

$$P_{q,ru}^{jk} = \rho_{ru}^{jk} + \nu_{P_q\,ru}^{\ jk}; \quad q=1,2\ldots$$

$$L_{q,ru}^{jk} = \rho_{ru}^{jk} + \lambda_q\, N_{q,ru}^{jk} + \nu_{L_q\,ru}^{\ jk}$$

To simplify notation, when different frequencies are considered, we will remove the subscript "ru".

$$P_q^{jk} = \rho^{jk} + \nu_{P_q}^{jk}; \quad q=1,2\ldots$$

$$L_q^{jk} = \rho^{jk} + \lambda_q\, N_q^{jk} + \nu_{L_q}^{jk}$$

K sat. in view → K 'SD'
→ $K(K\text{-}1)$ 'DD',
but only $K\text{-}1$ DDs are linearly independent

We assume the following measurement errors:

$$\sigma_{P_q} \approx 0.5\,\mathrm{m} \qquad \Rightarrow \qquad \sigma_{P_q^{jk}} \approx 1\,\mathrm{m}$$

$$\sigma_{L_q} \approx 0.5\,\mathrm{cm} \qquad\qquad \sigma_{L_q^{jk}} \approx 1\,\mathrm{cm}$$

Take the highest elevation as the reference satellite to minimize measurement error.

As commented before, the ambiguity terms are integer numbers, and we can take benefit of this property to fix such ambiguities applying integer ambiguity resolution techniques.

gAGE/UPC research group of Astronomy and Geomatics
BarcelonaTECH, Spain

Contents

Resolving ambiguities one at a Time

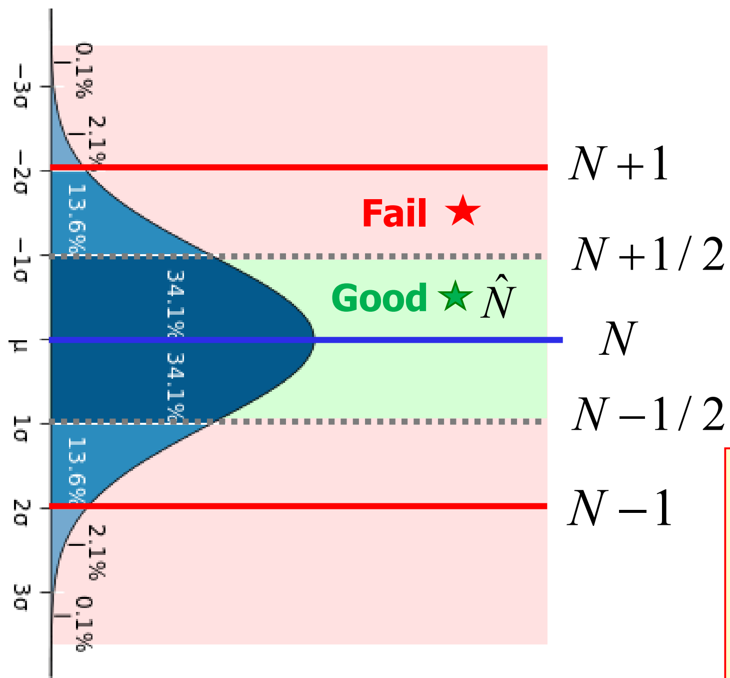
A simple trial would be (for instance using L1 and P1):

$$P_1^{jk} = \rho^{jk} + \nu_{P_1}^{jk}$$
$$L_1^{jk} = \rho^{jk} + \lambda_1 N_1^{jk} + \nu_{L_1}^{jk}$$

$$\rightarrow L_1^{jk} - P_1^{jk} = \lambda_1 N_1^{jk} + \nu_{P_1}^{jk} \rightarrow \hat{N}_1^{jk} = \left[\frac{L_1^{jk} - P_1^{jk}}{\lambda_1}\right]_{roundoff}$$

$$\lambda_1 \approx 20\,\text{cm}$$
$$\sigma_{P_1^{jk}} \approx 1\,\text{m}$$
$$\sigma_{L_1^{jk}} \approx 1\,\text{cm}$$

$$\sigma_{\hat{N}_1^{jk}} \approx \frac{1}{\lambda_1}\sigma_{P_1^{jk}} \approx 5$$

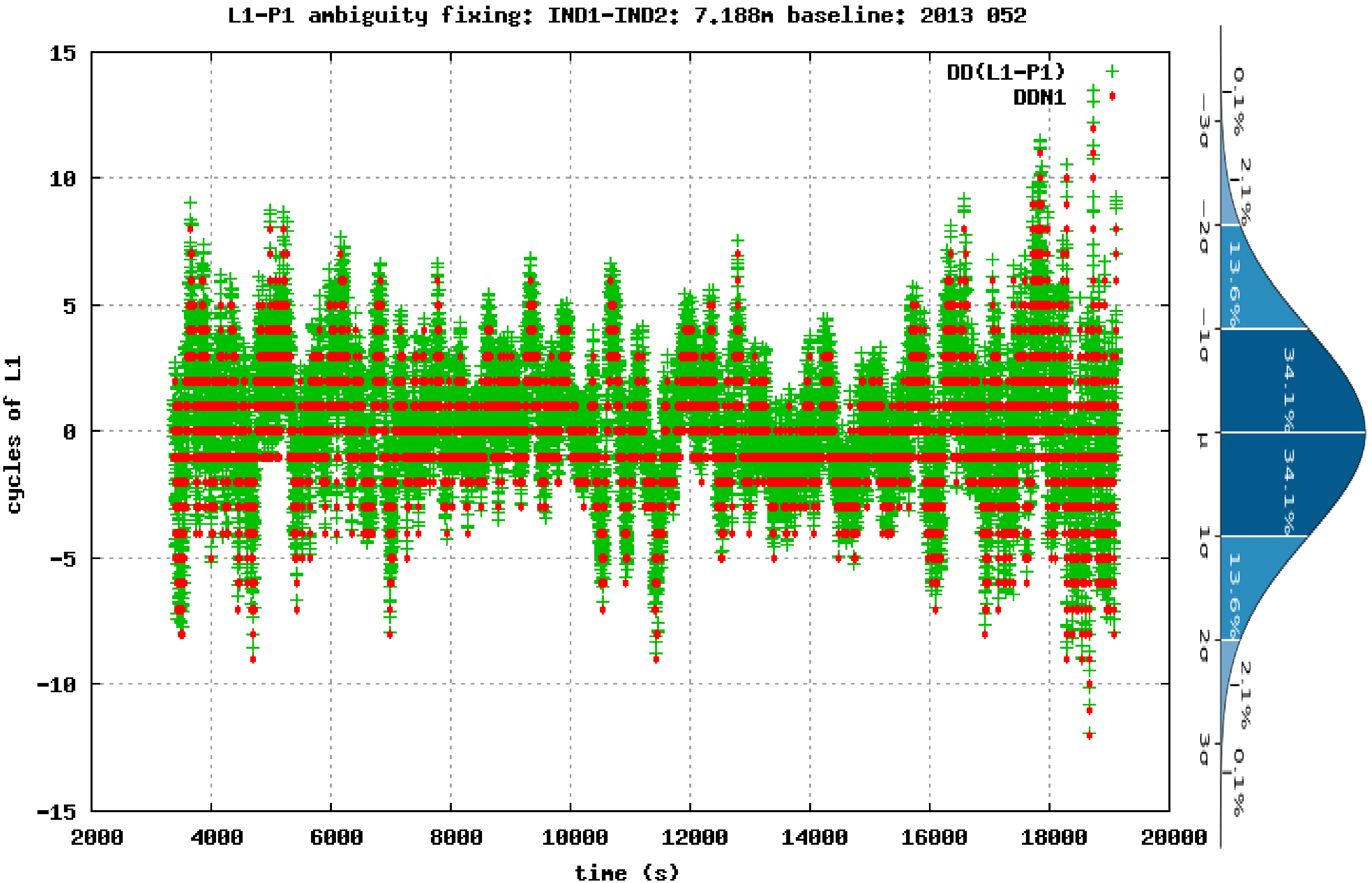


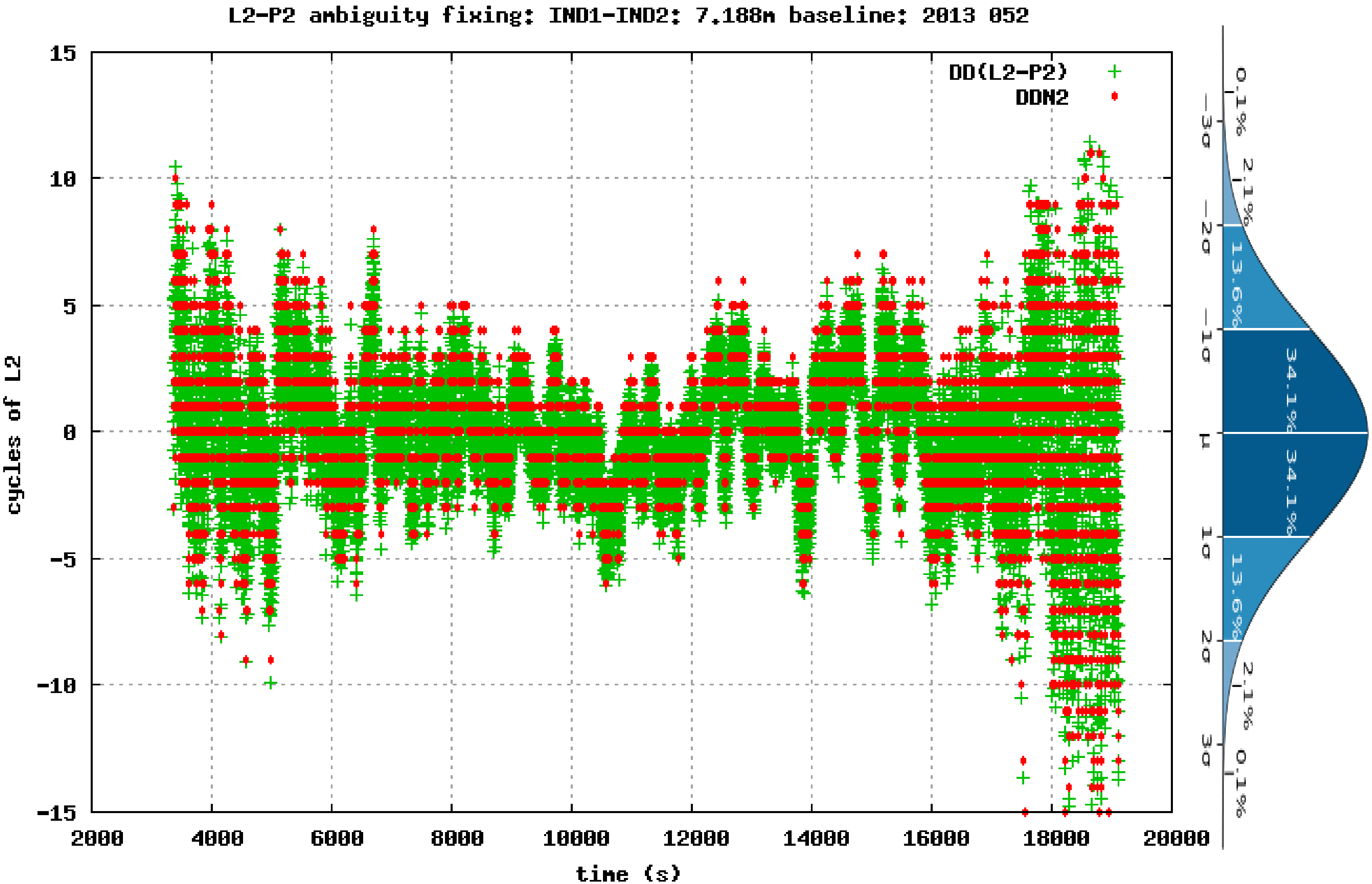
Too much error (5 wavelengths)!

Note that, assuming a Gaussian distribution of errors, $\sigma_{\hat{N}_1^{jk}} \approx 1/2$ guarantee only the 68% of success

Similar results with *L2*, *P2* measurements

As the ambiguity is constant (between cycle-slips), we would try to reduce uncertainty by averaging the estimate on time, but we will need 100 epochs to reduce noise up to ½ (but measurement errors are highly correlated on time!)


L1-P1 ambiguity fixing: IND1-IND2: 7.188m baseline: 2013 052
DD(L1-P1)
DDN1
cycles of L1
time (s)
15
10
5
0
-5
-10
-15
2000
4000
6000
8000
10000
12000
14000
16000
18000
20000
0.1%
2.1%
13.6%
34.1%
34.1%
13.6%
2.1%
0.1%
−3σ
−2σ
−1σ
μ
1σ
2σ
3σ


L2-P2 ambiguity fixing: IND1-IND2: 7.188m baseline: 2013 052
DD(L2-P2)
DDN2
cycles of L2
time (s)
15
10
5
0
-5
-10
-15
2000
4000
6000
8000
10000
12000
14000
16000
18000
20000
0.1%
2.1%
13.6%
34.1%
34.1%
13.6%
2.1%
0.1%
−3σ
−2σ
−1σ
μ
1σ
2σ
3σ

Contents

Ambiguity resolution Techniques

1. Resolving ambiguities one at a time
   - Single-frequency measurements
   - Dual-frequency measurements
   - Three-frequency measurements
2. Resolving ambiguities as a set: Search techniques
   - Least-Squares Ambiguity Search Technique.
   - LAMBDA Method.

Resolving ambiguities one at a Time

Dual frequency measurements: wide-laning with the Melbourne-Wübbena combination

$$P_1^{jk} = \rho^{jk} + \nu_{P_1}^{jk}$$
$$P_2^{jk} = \rho^{jk} + \nu_{P_2}^{jk}$$

$$L_1^{jk} = \rho^{jk} + \lambda_1 N_1^{jk} + \nu_{L_1}^{jk}$$
$$L_2^{jk} = \rho^{jk} + \lambda_2 N_2^{jk} + \nu_{L_2}^{jk}$$

$$P_N^{jk} = \frac{f_1 P_1^{jk} + f_2 P_2^{jk}}{f_1 + f_2} = \rho^{jk} + \nu_{P_N}^{jk}$$

$$L_W^{jk} = \frac{f_1 L_1^{jk} - f_2 L_2^{jk}}{f_1 - f_2} = \rho^{jk} + \lambda_W N_W^{jk} + \nu_{L_W}^{jk}$$

$$N_W = N_1 - N_2$$
$$\lambda_W = \frac{c}{f_1 - f_2} \approx 86.2\,cm$$

$$\sigma_{P_N^{jk}} \approx \sigma_{P_1^{jk}} / \sqrt{2} \approx 71\,cm$$
$$\sigma_{L_W^{jk}} \approx 6\,\sigma_{L_1^{jk}} \approx 6\,\text{cm}$$

$$L_W^{jk} - P_N^{jk} = \lambda_W N_W^{jk} + \nu_{P_N}^{jk} \rightarrow \hat{N}_W^{jk} = \left[\frac{L_W^{jk} - P_N^{jk}}{\lambda_W}\right]_{roundoff}$$

$$\sigma_{\hat{N}_W^{jk}} \approx \frac{1}{\lambda_W} \sigma_{P_N^{jk}} \approx \frac{71cm}{86.2cm} \approx 0.8$$

Now, with uncorrelated measurements from 10 epochs will reduce noise up to about ¼.

Fixing N_1 (after fixing N_W)

$$L_1^{jk} - L_2^{jk} = \lambda_1 N_1^{jk} - \lambda_2 N_2^{jk} + \nu_{L_1-L_2}^{jk}$$
$$= (\lambda_1 - \lambda_2) N_1^{jk} + \lambda_2 N_W^{jk} + \nu_{L_1-L_2}^{jk}$$

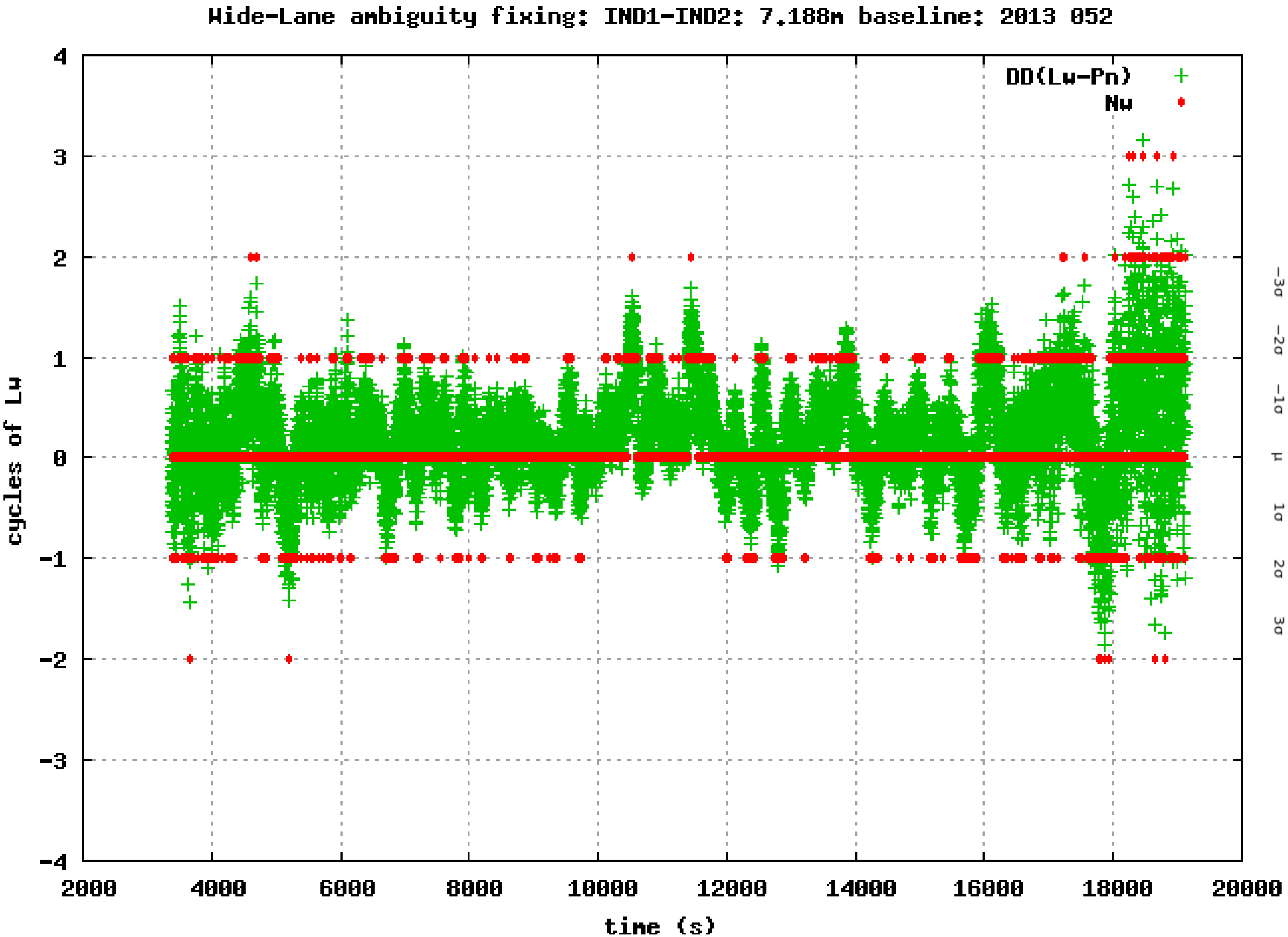
$$\lambda_1 = 19.0\,\text{cm}$$
$$\lambda_2 = 24.4\,\text{cm}$$
$$\lambda_2 - \lambda_1 = 5.4\,\text{cm}$$
$$\sigma_{L_1^{jk}} \approx 1\,\text{cm}$$

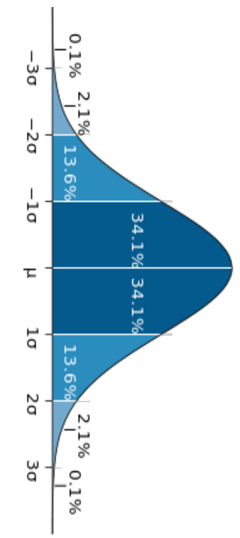
$$\hat{N}_1^{jk} = \left[\frac{L_1^{jk} - L_2^{jk} - \lambda_2 \hat{N}_W^{jk}}{\lambda_1 - \lambda_2}\right]_{roundoff}$$

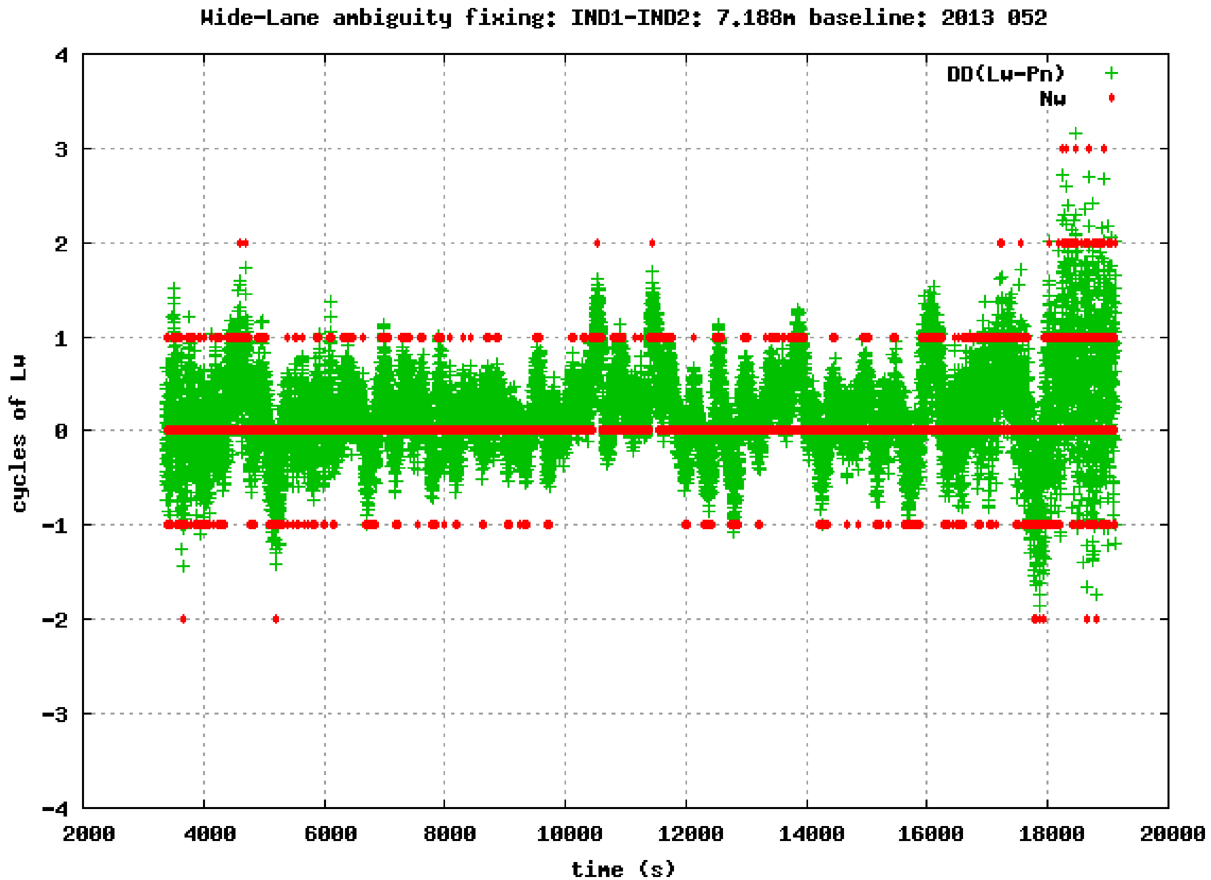
$$\hat{N}_2^{jk} = \hat{N}_1^{jk} - \hat{N}_W^{jk}$$

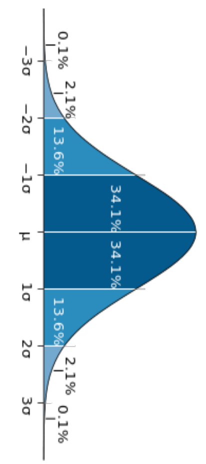
$$\sigma_{\hat{N}_1^{jk}} \approx \frac{1}{\lambda_1 - \lambda_2} \sqrt{2}\,\sigma_{L_1^{jk}} \approx \frac{1.4cm}{5.4cm} \approx 1/4$$

gAGE/UPC research group of Astronomy and Geomatics
BarcelonaTECH, Spain





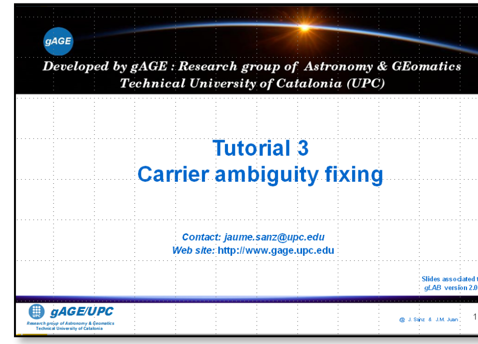


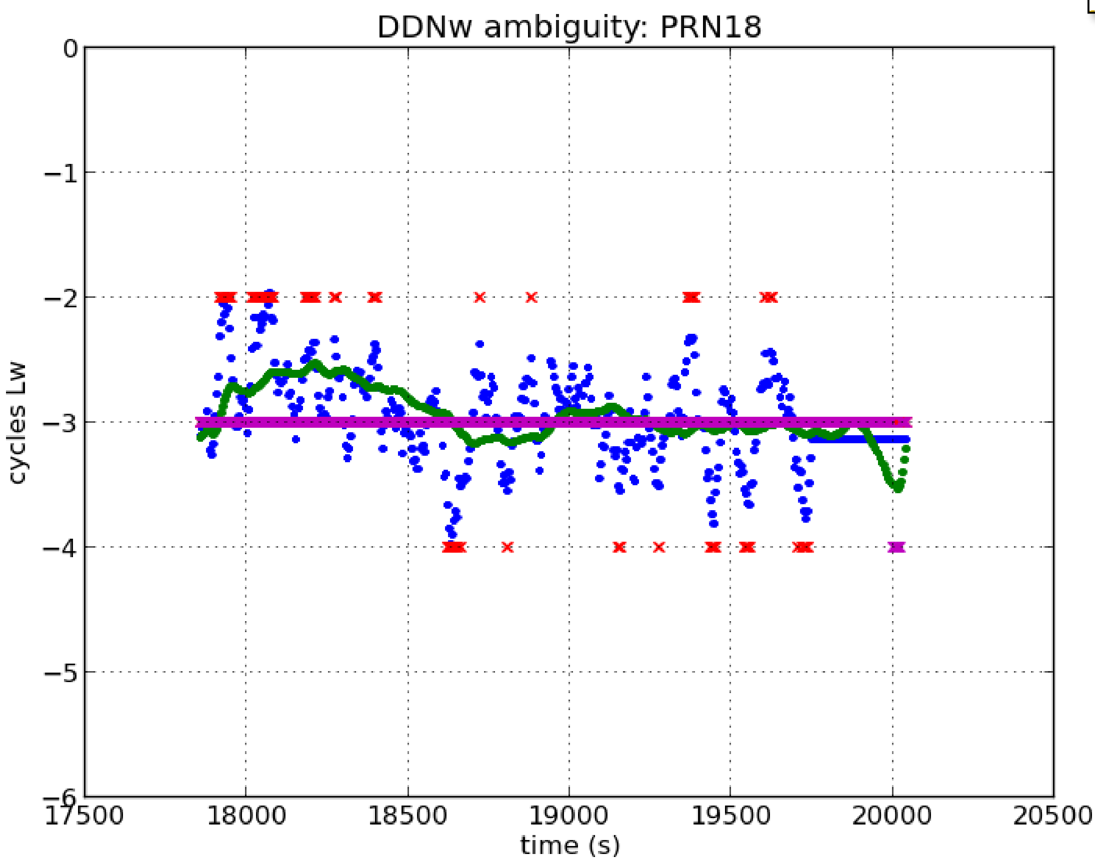


Once the integer ambiguities are known, the carrier phase measurements become unambiguous pseudoranges, accurate at the centimetre level (in DD), or better.

Thence, the estimation of the relative position vector is straightforward following the same approach as with pseudoranges.

300 seconds smoothing





Exercises:

1) Consider the wide-lane combination of carrier phase measurements

$L_W = \frac{f_1 L_1 - f_2 L_2}{f_1 - f_2}$, where L_W is given in length units (i.e. $L_i = \lambda_i \phi_i$).

Show that the corresponding wavelength is: $\lambda_W = \frac{c}{f_1 - f_2}$

Hint:

$L_W = \lambda_W \phi_W$; $\phi_W = \phi_1 - \phi_2$

2) Assuming L_1, L_2 uncorrelated measurements with equal noise σ_L, show that:

$$\sigma_{L_W} = \frac{\sqrt{\gamma_{12}+1}}{\sqrt{\gamma_{12}}-1}\sigma_L \quad ; \quad \gamma_{12} = \left(\frac{f_1}{f_2}\right)^2$$

Contents

Ambiguity resolution Techniques

1. Resolving ambiguities one at a time
   - Single-frequency measurements
   - Dual-frequency measurements
   - Three-frequency measurements
2 . Resolving ambiguities as a set: Search techniques
   - Least-Squares Ambiguity Search Technique.
   - LAMBDA Method.

gAGE

Three Frequency measurements:

We still consider the above problem of relative positioning in DD for short baselines (e.g. < 10 km) ➔ *Ionosphere, troposphere and wind-up differential errors cancel.*

GPS	Frequency	Wavelengths	Combinations
L1	154 x 10.23 MHz	λ_1 =0.190 m	$\lambda_2 - \lambda_1$ =0.054 m
L2	120 x 10.23 MHz	λ_2 =0.244 m	λ_W = 0.862 m
L5	115 x 10.23 MHz	λ_5 =0.255 m	λ_{EW} =5.861 m

With three frequency systems, having two close frequencies it is possible to generate an extra-wide-lane signal to enable the single epoch ambiguity fixing.

We drop here the superscript (*jk*) for simplicity

$$\lambda_i = \frac{c}{f_i}; \quad \lambda_W = \frac{c}{f_1 - f_2}; \quad \lambda_{EW} = \frac{c}{f_2 - f_5}$$

$$L_i = \rho + \lambda_i N_i + \nu_{L_i} \;; i = 1,2,5$$

$$L_W = \frac{f_1 L_1 - f_2 L_2}{f_1 - f_2} = \rho + \lambda_W N_W + \nu_{L_W}$$

$$L_{EW} = \frac{f_2 L_2 - f_5 L_5}{f_2 - f_5} = \rho + \lambda_{EW} N_{EW} + \nu_{L_{EW}}$$

$$P_N = \frac{f_1 P_1 + f_2 P_2}{f_1 + f_2} = \rho + \nu_{P_N}$$

$$P_{EN} = \frac{f_2 P_2 + f_5 P_5}{f_2 + f_5} = \rho + \nu_{P_{EN}}$$

$$\sigma_{L_W} = \frac{\sqrt{\gamma_{12}+1}}{\sqrt{\gamma_{12}}-1}\sigma_{L_1} \approx 5,7\,cm$$

$$\sigma_{L_{EW}} = \frac{\sqrt{\gamma_{25}+1}}{\sqrt{\gamma_{25}}-1}\sigma_{L_1} \approx 33,3\,\mathrm{cm}$$

$$\sigma_{P_N} = \frac{\sqrt{\gamma_{12}+1}}{\sqrt{\gamma_{12}}+1}\sigma_{P_1} \approx 0,712\,\mathrm{m}$$

$$\sigma_{P_{EN}} = \frac{\sqrt{\gamma_{25}+1}}{\sqrt{\gamma_{25}}+1}\sigma_{P_1} \approx 0,707\,\mathrm{m}$$

$$\gamma_{12} = \left(f_1 / f_2\right)^2 = (77/60)^2$$

$$\gamma_{25} = \left(f_2 / f_5\right)^2 = (24/23)^2$$

$$N_W = N_1 - N_2$$

$$N_{EW} = N_2 - N_5$$

Exercise:
Justify the previous expressions for σ.

gAGE/UPC research group of Astronomy and Geomatics
Barcelona**TECH**, Spain

We still consider the above problem of relative positioning in DD for short baselines (e.g. < 10 km) ➔ *Ionosphere, troposphere and wind-up differential errors cancel .*

GPS	Frequency	Wavelengths	Combinations
L1	154 x 10.23 MHz	λ_1 =0.190 m	$\lambda_2 - \lambda_1$ =0.054 m
L2	120 x 10.23 MHz	λ_2 =0.244 m	λ_W = 0.862 m
L5	115 x 10.23 MHz	λ_5 =0.255 m	λ_{EW} =5.861 m

$$L_1 = \rho + \lambda_1 N_1 + \nu_{L_1}$$
$$L_2 = \rho + \lambda_2 N_2 + \nu_{L_2}$$

$$\sigma_{L_1} \approx \sigma_{L_2} \approx 1\,\text{cm}$$
$$\sigma_{P_1} \approx \sigma_{P_2} \approx 1\,\text{m}$$

$$L_W = \rho + \lambda_W N_W + \nu_{L_W} \qquad P_N = \rho + \nu_{P_N}$$
$$L_{EW} = \rho + \lambda_{EW} N_{EW} + \nu_{L_{EW}} \qquad P_{EN} = \rho + \nu_{P_{EN}}$$

$$N_W = N_1 - N_2 \quad ; \quad N_{EW} = N_2 - N_5$$

$$\hat{N}_{EW} = \left[\frac{L_{EW} - P_{EN}}{\lambda_{EW}} \right]_{roundoff} \longleftrightarrow \sigma_{\hat{N}_{EW}} \approx \frac{1}{\lambda_{EW}} \sigma_{P_{EN}} \approx \frac{0.71\,\text{m}}{5.861\,\text{m}} \approx 0.12$$

$$\hat{N}_W = \left[\frac{\lambda_{EW} \hat{N}_{EW} - \left(L_{EW} - L_W\right)}{\lambda_W} \right]_{roundoff} \longleftrightarrow \sigma_{\hat{N}_W} \approx \frac{1}{\lambda_W} \sigma_{L_{EW}} \approx \frac{33.3\,\text{cm}}{86.2\,\text{cm}} \approx 0.39$$

$$\hat{N}_1 = \left[\frac{L_1 - L_2 - \lambda_2 \hat{N}_W}{\lambda_1 - \lambda_2} \right]_{roundoff} \longleftrightarrow \sigma_{\hat{N}_1} \approx \frac{1}{\lambda_1 - \lambda_2} \sqrt{2} \sigma_{L_1} \approx \frac{1.4cm}{5.4cm} \approx 1/4$$

$$\gamma_{12} = \left(f_1 / f_2\right)^2 = (77/60)^2$$
$$\gamma_{25} = \left(f_2 / f_5\right)^2 = (24/23)^2$$

$$\sigma_{P_N} = \frac{\sqrt{\gamma_{12}+1}}{\sqrt{\gamma_{12}}+1} \sigma_{P_1} \approx 0,71\,\text{m}$$
$$\sigma_{P_{EN}} = \frac{\sqrt{\gamma_{25}+1}}{\sqrt{\gamma_{25}}+1} \sigma_{P_1} \approx 0,71\,\text{m}$$

$$\sigma_{L_W} = \frac{\sqrt{\gamma_{12}+1}}{\sqrt{\gamma_{12}}-1} \sigma_{L_1} \approx 5,7\,cm$$
$$\sigma_{L_{EW}} = \frac{\sqrt{\gamma_{25}+1}}{\sqrt{\gamma_{25}}-1} \sigma_{L_1} \approx 33,3\,cm$$

Exercise:

Repeat the previous study for the Galileo signals E1, E5b and E5a

Galileo	Frequency	Wavelengths	Combinations
E1	154 x 10.23 MHz	λ_1 =0.190 m	$\lambda_2 - \lambda_1$ =0.058 m
E5b	118 x 10.23 MHz	λ_2 =0.248 m	λ_W = 0.814 m
E5a	115 x 10.23 MHz	λ_3 =0.255 m	λ_{EW} =9.768 m

$$L_1 = \rho + \lambda_1 N_1 + \nu_{L_1}$$
$$L_2 = \rho + \lambda_2 N_2 + \nu_{L_2}$$

$$\sigma_{L_1} \approx \sigma_{L_2} \approx 1\,\mathrm{cm}$$
$$\sigma_{P_1} \approx \sigma_{P_2} \approx 1\,\mathrm{m}$$

$$L_W = \rho + \lambda_W N_W + \nu_{L_W} \qquad P_N = \rho + \nu_{P_N}$$
$$L_{EW} = \rho + \lambda_{EW} N_{EW} + \nu_{L_{EW}} \qquad P_{EN} = \rho + \nu_{P_{EN}}$$

$$\hat{N}_{EW} = \left[\frac{L_{EW} - P_{EN}}{\lambda_{EW}}\right]_{roundoff} \longleftrightarrow \sigma_{\hat{N}_{EW}} \approx \frac{1}{\lambda_{EW}} \sigma_{P_{EN}} \approx [\qquad]$$

$$\hat{N}_W = \left[\frac{\lambda_{EW} \hat{N}_{EW} - (L_{EW} - L_W)}{\lambda_W}\right]_{roundoff} \longleftrightarrow \sigma_{\hat{N}_W} \approx \frac{1}{\lambda_W} \sigma_{L_{EW}} \approx [\qquad]$$

$$\hat{N}_1 = \left[\frac{L_1 - L_2 - \lambda_2 \hat{N}_W}{\lambda_1 - \lambda_2}\right]_{roundoff} \longleftrightarrow \sigma_{\hat{N}_1} \approx \frac{1}{\lambda_1 - \lambda_2} \sqrt{2} \sigma_{L_1} \approx [\qquad]$$

$$\gamma_{12} = (f_1 / f_2)^2 = (77/59)^2$$
$$\gamma_{23} = (f_2 / f_3)^2 = (118/115)^2$$

$$\sigma_{P_N} = \frac{\sqrt{\gamma_{12} + 1}}{\sqrt{\gamma_{12}} + 1} \sigma_{P_1} \approx [\qquad]$$
$$\sigma_{P_{EN}} = \frac{\sqrt{\gamma_{25} + 1}}{\sqrt{\gamma_{25}} + 1} \sigma_{P_1} \approx [\qquad]$$

$$\sigma_{L_W} = \frac{\sqrt{\gamma_{12} + 1}}{\sqrt{\gamma_{12}} - 1} \sigma_{L_1} \approx [\qquad]$$
$$\sigma_{L_{EW}} = \frac{\sqrt{\gamma_{25} + 1}}{\sqrt{\gamma_{25}} - 1} \sigma_{L_1} \approx [\qquad]$$

Exercise:

Repeat the previous study for the Galileo signals E1, E5b and E5a

Galileo	Frequency	Wavelengths	Combinations
E1	154 x 10.23 MHz	λ_1 =0.190 m	$\lambda_2-\lambda_1$ =0.058 m
E5b	118 x 10.23 MHz	λ_2 =0.248 m	λ_W = 0.814 m
E5a	115 x 10.23 MHz	λ_3 =0.255 m	λ_{EW} =9.768 m

$$L_1 = \rho + \lambda_1 N_1 + \nu_{L_1}$$
$$L_2 = \rho + \lambda_2 N_2 + \nu_{L_2}$$

$$\sigma_{L_1} \approx \sigma_{L_2} \approx 1\,\mathrm{cm}$$
$$\sigma_{P_1} \approx \sigma_{P_2} \approx 1\,\mathrm{m}$$

$$L_W = \rho + \lambda_W N_W + \nu_{L_W} \qquad P_N = \rho + \nu_{P_N}$$
$$L_{EW} = \rho + \lambda_{EW} N_{EW} + \nu_{L_{EW}} \qquad P_{EN} = \rho + \nu_{P_{EN}}$$

$$\hat{N}_{EW} = \left[\frac{L_{EW} - P_{EN}}{\lambda_{EW}}\right]_{roundoff} \longleftrightarrow \sigma_{\hat{N}_{EW}} \approx \frac{1}{\lambda_{EW}}\sigma_{P_{EN}} \approx \frac{0.71\,\mathrm{m}}{9.768\,\mathrm{m}} \approx 0.07$$

$$\hat{N}_W = \left[\frac{\lambda_{EW}\hat{N}_{EW} - \left(L_{EW} - L_W\right)}{\lambda_W}\right]_{roundoff} \longleftrightarrow \sigma_{\hat{N}_W} \approx \frac{1}{\lambda_W}\sigma_{L_{EW}} \approx \frac{54.9\,\mathrm{cm}}{81.4\,\mathrm{cm}} \approx 0.67$$

$$\hat{N}_1 = \left[\frac{L_1 - L_2 - \lambda_2 \hat{N}_W}{\lambda_1 - \lambda_2}\right]_{roundoff} \longleftrightarrow \sigma_{\hat{N}_1} \approx \frac{1}{\lambda_1 - \lambda_2}\sqrt{2}\sigma_{L_1} \approx \frac{1.4\,\mathrm{cm}}{5.8\,\mathrm{cm}} \approx 1/4$$

$$\gamma_{12} = \left(f_1/f_2\right)^2 = (77/59)^2$$
$$\gamma_{23} = \left(f_2/f_3\right)^2 = (118/115)^2$$

$$\sigma_{P_N} = \frac{\sqrt{\gamma_{12}+1}}{\sqrt{\gamma_{12}}+1}\sigma_{P_1} \approx 0,71\,\mathrm{m}$$
$$\sigma_{P_{EN}} = \frac{\sqrt{\gamma_{25}+1}}{\sqrt{\gamma_{25}}+1}\sigma_{P_1} \approx 0,71\,\mathrm{m}$$

$$\sigma_{L_W} = \frac{\sqrt{\gamma_{12}+1}}{\sqrt{\gamma_{12}}-1}\sigma_{L_1} \approx 5.4\,\mathrm{cm}$$
$$\sigma_{L_{EW}} = \frac{\sqrt{\gamma_{25}+1}}{\sqrt{\gamma_{25}}-1}\sigma_{L_1} \approx 54.9\,\mathrm{cm}$$

Contents

Ambiguity resolution Techniques

1. Resolving ambiguities one at a time
   - Single-frequency measurements
   - Dual-frequency measurements
   - Three-frequency measurements
2. Resolving ambiguities as a set: Search techniques
   - Least-Squares Ambiguity Search Technique.
   - LAMBDA Method.

Resolving Ambiguities as a set

As a driven problem to study the ambiguity fixing, we will consider problem of differential positioning in DD for short baselines (e.g. < *10* km). To simplify, we will consider only carrier measurements at a single or dual frequency.

$$\begin{cases} L_q^{12}(t_i) = \rho^{12}(t_i) + N_q^{12} + \upsilon_{L_q}^{12}(t_i) \\ L_q^{13}(t_i) = \rho^{13}(t_i) + N_q^{13} + \upsilon_{L_q}^{13}(t_i) \\ \vdots \\ L_q^{1K-1}(t_i) = \rho^{1K-1}(t_i) + N_q^{1K-1} + \upsilon_{L_q}^{1K-1}(t_i) \end{cases} \qquad q = 1, 2 \ldots$$

Static position	Equations	Unknowns
Single frequency	$(K-1) \cdot n_t$	3+(K-1)
Dual frequency	$2(K-1) \cdot n_t$	3+2(K-1)

Kin. position	Equations	Unknowns
Single frequency	$(K-1) \cdot n_t$	$3 \cdot n_t+(K-1)$
Dual frequency	$2(K-1) \cdot n_t$	$3 \cdot n_t+2(K-1)$

$K \geq 4,\ n_t \geq 2$ $\quad$ $K \geq 5,\ n_t \geq 4$

In principle, the estimation of ambiguities in this system is not a big problem **if we can wait enough time and the unmodelled errors are not so large**.

Each epoch brings a set of $(K\text{-}1)$ DD (i.e. equations) for each frequency.
Note: n_t is the number of epochs

Linear Model:

$$L_q^{jk}(t_i) = \rho^{jk}(t_i) + N_q^{jk} + \upsilon_{L_q}^{jk}(t_i)$$

$$\rho^{jk}(t_i) = \rho_0^{jk}(t_i) - \hat{\boldsymbol{\rho}}_0^{jk}(t_i) \cdot \Delta\mathbf{r}(t_i)$$

$$L^{jk}(t_i) - \rho_0^{jk}(t_i) = -\hat{\boldsymbol{\rho}}_0^{jk}(t_i) \cdot \Delta\mathbf{r}_{ru}(t_i) + \lambda N^{jk} + \nu_L^{jk}(t_i)$$

Prefit-residual *y(t)* $\qquad$ *G(t)*

We can estimate all parameters (position and ambiguities) as a set by considering the **over-dimensioned system** of linear equations and solving it by the LS criterion.

$$\mathbf{y}(t_i) = \mathbf{G}(t_i)\, \Delta\mathbf{r}(t_i) + \lambda\mathbf{N} + \boldsymbol{\nu}$$

Resolving Ambiguities as a set

$$\mathbf{y}(t_i) = \mathbf{G}(t_i)\,\Delta\mathbf{r}(t_i) + \lambda\mathbf{N} + \mathbf{v}(t_i)$$

K vector — K x 3 matrix — 3 vector — K vector

Single Freq: K=K-1
Dual Freq. : K=2(K-1)

For static positioning, considering two epochs (for instance):

$$\begin{bmatrix}\mathbf{y}(t_i)\\ \mathbf{y}(t_{i+1})\end{bmatrix} = \begin{bmatrix}\mathbf{G}(t_i)\\ \mathbf{G}(t_{i+1})\end{bmatrix}\Delta\mathbf{r} + \lambda\begin{bmatrix}\mathbf{I}\\ \mathbf{I}\end{bmatrix}\mathbf{N} + \begin{bmatrix}\mathbf{v}(t_i)\\ \mathbf{v}(t_{i+1})\end{bmatrix}$$

In general, mixing several epochs, we will write:

$$\mathbf{y} = \mathbf{G}\,\Delta\mathbf{r} + \lambda\,\mathbf{A}\,\mathbf{N} + \mathbf{v}$$

Using the least-squares criterion, we can look for a real valued 3-vector $\Delta\mathbf{r}$ and a K-vector of integers $\mathbf{N}$ that minimizes the cost function (sum of squared residuals):

$$c(\Delta\mathbf{r},\mathbf{N}) = \left\|\mathbf{y} - \mathbf{G}\,\Delta\mathbf{r} + \lambda\,\mathbf{A}\,\mathbf{N}\right\|$$

Weighted norm can be taken as well

The problem can be easily reformulated for the kinematic case. Kalman filtering can be applied as well.

Resolving Ambiguities as a set

Different strategies can be applied:

- **To Float the ambiguities** (i.e. treating the ambiguities as real numbers).
- **To Search ambiguities** over a limited set of integers to 'find the best solution'.
- **To solve as an Integer Least-Squares problem**.

For an observation span relatively long, e.g. one hour, the floated ambiguities would typically be very close to integers, and the change in the position solution from the float to the fixed solution should not be large.

As the observation span becomes smaller, ambiguity resolution play a more important role. But very short observation spans implies the risk of wrong ambiguity fixing, which can degrade the position solution significantly.

The performance, is thence measured by:

1. Initialization time
2. Reliability (or, correctness) of the integer estimates

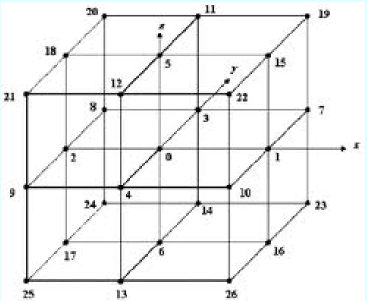
Contents

Ambiguity resolution Techniques

1. Resolving ambiguities one at a time
   - Single-frequency measurements
   - Dual-frequency measurements
   - Three-frequency measurements

2 . Resolving ambiguities as a set: Search techniques
   - Least-Squares Ambiguity Search Technique.
   - LAMBDA Method.

Search techniques



Strategy:

- Define a volume to be searched
- Set up a grid within this volume
- Define a cost function (e.g. the sum of squared residuals)
- Evaluate the cost function at each grid point

Solution corresponds to the grid point with the lowest value of the cost function

Position domain

Ambiguity Function Method (AFM)

ARCE

…………

Ambiguity domain

LSAS (Hatch, 1990)

LAMBDA (Teunissen, 1993)

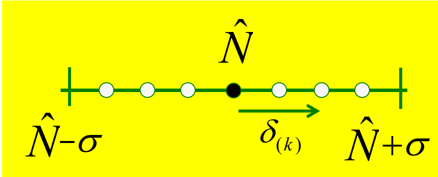
MLAMBDA (Chang et al. 2005)

OMEGA (Kim and Langley, 2000)

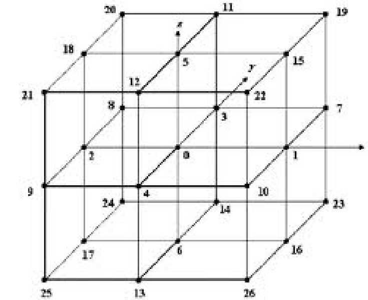
FASF (Chen and Lachappelle, 1995)

IP (Xu et al., 1995)

……………



Search techniques



A conceptually simpler approach would consist on:

- Estimate the floated solution $\hat{N}$ and its uncertainty (e.g. $\hat{N}$=2502347.74 cycles, $\sigma_{\hat{N}} = 0.6$ cycles)
- Define as a volume to be searched $\left(\text{e.g. } \pm 3\sigma_{\hat{N}} \approx \pm 2 \text{ cycles}\right)$ and evaluate the cost function (the RMS residuals) over the 6 ambig.: $2502345, \cdots, 2502350$

The previous search must be done for each satellite in view.

- If there are 5 satellites tracked ➔ 4 DD ambiguities ➔ 6^4 = 1 296 combinations
- If there are 8 satellites tracked ➔ 7 DD ambiguities ➔ 6^7 =279 376 combinations

The integer ambiguity solution corresponding to **the smallest RMS residuals is used to select the candidate**. However if two or more candidates give roughly similar values of RMS, the test can not be resolute.

➔A ratio test (of 2 or 3, depending of the algorithm) between the two smallest RMS is often used to validate the test.
If the ratio is under these values, no integer solution can be determined and is better to use the floated solution.

Contents

Ambiguity resolution Techniques

LAMBDA Method

Consider again the previous problem of estimating $\Delta\mathbf{r}$, a *3*-vector of real numbers, and $\mathbf{N}$ a *(K-1)*-vector of integers, which are solution of

$$\mathbf{y} = \mathbf{G}\,\Delta\mathbf{r} + \lambda\,\mathbf{A}\,\mathbf{N} + \mathbf{v} \qquad \min\left\|\mathbf{y} - \mathbf{G}\,\Delta\mathbf{r} - \lambda\,\mathbf{A}\,\mathbf{N}\right\|_{\mathbf{W_y}}$$

To better exploit the internal correlations [*], we consider now the covariance $\mathbf{W_y} = \mathbf{P_y}^{-1}$

Let be the <u>float solution and covariance matrix</u>: $\begin{bmatrix}\Delta\hat{\mathbf{r}}\\ \hat{\mathbf{N}}\end{bmatrix}$; $\mathrm{Cov}\begin{bmatrix}\Delta\hat{\mathbf{r}}\\ \hat{\mathbf{N}}\end{bmatrix} = \begin{bmatrix}\mathbf{P}_{\Delta\hat{\mathbf{r}}} & \mathbf{P}_{\Delta\hat{\mathbf{r}},\hat{\mathbf{N}}}\\ \mathbf{P}_{\Delta\hat{\mathbf{r}},\hat{\mathbf{N}}} & \mathbf{P}_{\hat{\mathbf{N}}}\end{bmatrix}$

It can be shown the following orthogonal decomposition:

$$\left\|\mathbf{y} - \mathbf{G}\,\Delta\mathbf{r} - \lambda\,\mathbf{A}\,\mathbf{N}\right\|^2_{\mathbf{W_y}} = \underbrace{\left\|\mathbf{y} - \mathbf{G}\,\Delta\hat{\mathbf{r}} - \lambda\,\mathbf{A}\,\hat{\mathbf{N}}\right\|^2_{\mathbf{W_y}}}_{\text{Residual of real-valued floated solution } (\Delta\hat{\mathbf{r}},\,\hat{\mathbf{N}})} + \left\|\Delta\mathbf{r} - \Delta\hat{\mathbf{r}}(\mathbf{N})\right\|^2_{\mathbf{W}_{\Delta\hat{\mathbf{r}}(\mathbf{N})}} + \lambda^2\left\|\mathbf{N} - \hat{\mathbf{N}}\right\|^2_{\mathbf{W}_{\hat{\mathbf{N}}}}$$

Residual of real-valued floated solution $\left(\Delta\hat{\mathbf{r}},\,\hat{\mathbf{N}}\right)$

---

[*] Remember that DD measurements are correlated, as already seen.

LAMBDA Method

Thence, we have to find $\Delta\mathbf{r}$ a *3*-vector of real numbers, and $\mathbf{N}$ a (*K-1*)-vector of integers minimizing:

$$\left\|\mathbf{y}-\mathbf{G}\,\Delta\mathbf{r}-\lambda\,\mathbf{A}\,\mathbf{N}\right\|^2_{\mathbf{W_y}} = \left\|\mathbf{y}-\mathbf{G}\,\Delta\hat{\mathbf{r}}-\lambda\,\mathbf{A}\,\hat{\mathbf{N}}\right\|^2_{\mathbf{W_y}} + \left\|\Delta\mathbf{r}-\Delta\hat{\mathbf{r}}(\mathbf{N})\right\|^2_{\mathbf{W}_{\Delta\hat{\mathbf{r}}(\mathbf{N})}} + \lambda^2\left\|\mathbf{N}-\hat{\mathbf{N}}\right\|^2_{\mathbf{W}_{\hat{\mathbf{N}}}}$$

This term is irrelevant for minimization since it does not depend on $\Delta\mathbf{r}$ and $\mathbf{N}$

This term can be made zero for any $\mathbf{N}$

This term must be minimized over the integers

Float solution and covariance matrix:

$$\begin{bmatrix}\Delta\hat{\mathbf{r}}\\ \hat{\mathbf{N}}\end{bmatrix} \quad ; \quad \mathrm{Cov}\begin{bmatrix}\Delta\hat{\mathbf{r}}\\ \hat{\mathbf{N}}\end{bmatrix} = \begin{bmatrix}\mathbf{P}_{\Delta\hat{\mathbf{r}}} & \mathbf{P}_{\Delta\hat{\mathbf{r}},\hat{\mathbf{N}}}\\ \mathbf{P}_{\Delta\hat{\mathbf{r}},\hat{\mathbf{N}}} & \mathbf{P}_{\hat{\mathbf{N}}}\end{bmatrix}$$

$$\mathbf{W}^{-1}_{\hat{\mathbf{N}}} = \mathbf{P}_{\hat{\mathbf{N}}} \qquad \mathbf{W}_{\Delta\hat{\mathbf{r}},\hat{\mathbf{N}}} = \mathbf{P}^{-1}_{\Delta\hat{\mathbf{r}},\hat{\mathbf{N}}}$$

$$\min\left\|\mathbf{N}-\hat{\mathbf{N}}\right\|^2_{\mathbf{W}_{\hat{\mathbf{N}}}} \rightarrow \mathbf{N}$$

$$\Delta\mathbf{r} = \Delta\hat{\mathbf{r}}(\mathbf{N}) = \Delta\hat{\mathbf{r}} - \mathbf{W}_{\Delta\hat{\mathbf{r}},\hat{\mathbf{N}}}\mathbf{W}^{-1}_{\hat{\mathbf{N}}}\left(\mathbf{N}-\hat{\mathbf{N}}\right)$$

The vectors $\Delta\mathbf{r}$ and $\mathbf{N}$ are often referred to as the **fixed user solution and fixed ambiguity**.

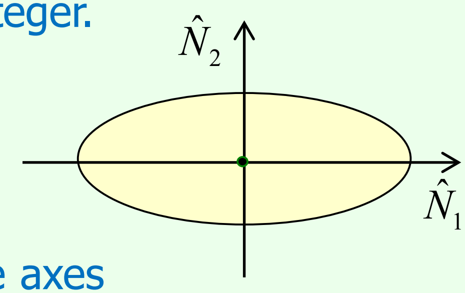
LAMBDA Method

The integer search: Finding the integer vector N that minimizes the cost function

$$c(\mathbf{N}) = \left\| \mathbf{N} - \hat{\mathbf{N}} \right\|^2_{\mathbf{W}_{\hat{\mathbf{N}}}} = \left(\mathbf{N} - \hat{\mathbf{N}}\right)^T \mathbf{W}_{\hat{\mathbf{N}}} \left(\mathbf{N} - \hat{\mathbf{N}}\right) \qquad \mathbf{W}_{\hat{\mathbf{N}}} = \mathbf{P}^{-1}_{\hat{\mathbf{N}}}$$

- A diagonal $\mathbf{W}_{\mathbf{N}}$ matrix would mean that the integer ambiguity estimates are uncorrelated.
- If the weight $\mathbf{W}_{\mathbf{N}}$ matrix is diagonal, the minimizing of the cost function is trivial. The best estimate is the float ambiguity rounded to the nearest integer.

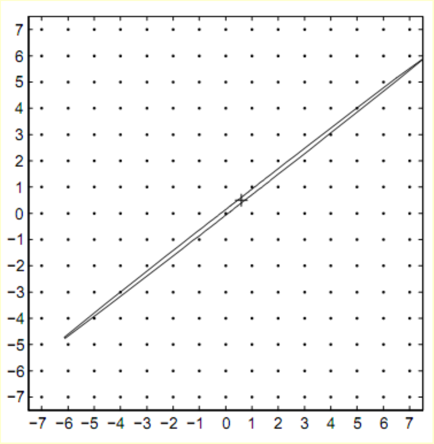
$$\mathbf{W}_{\hat{\mathbf{N}}} = \begin{bmatrix} 1/\sigma^2_{\hat{N}_1\hat{N}_1} & 0 \\ 0 & 1/\sigma^2_{\hat{N}_2\hat{N}_2} \end{bmatrix} \qquad c(\mathbf{N}) = \frac{\left(N_1 - \hat{N}_1\right)^2}{\sigma^2_{\hat{N}_1\hat{N}_1}} + \frac{\left(N_2 - \hat{N}_2\right)^2}{\sigma^2_{\hat{N}_2\hat{N}_2}}$$



Ellipse parallel to coordinate axes

In practice, the estimated (float) ambiguities are highly correlated and the ellipsoidal region stretches over a wide range of cycles. This is specially the case when the measurements are limited to a single epoch or only a few epochs.

Thence, points that appears much further away from the floated solution may have lower values of cost function than those which appear nearby. In this context, the search for integer vectors can by extremely inefficient.



To improve the computational efficiency of the search, the float ambiguities can be transformed so that the elongated ellipsoid turns into a sphere-like. Thus, the search can be limited to the neighbours of the floated ambiguity.

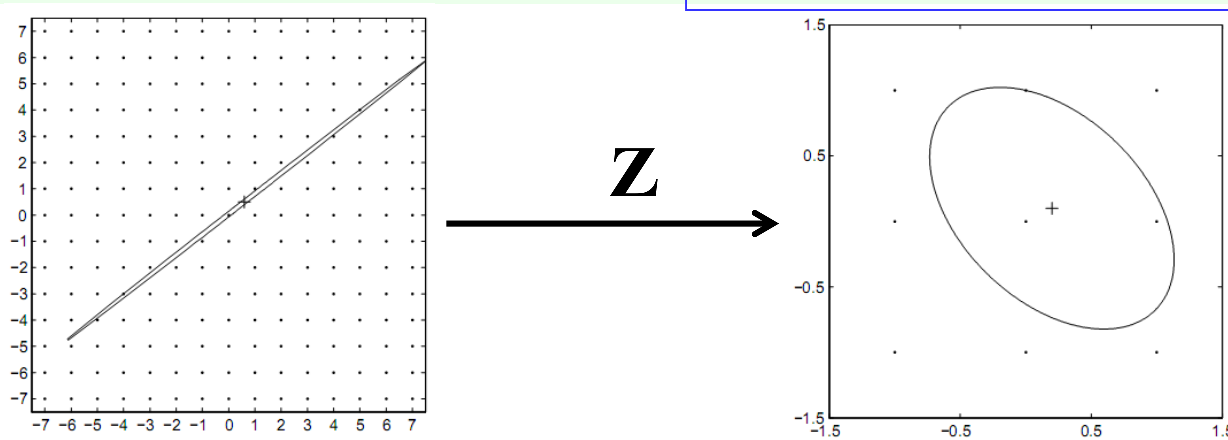
The idea could be to apply a transformation that decorrelates the ambiguities so that the matrix $\mathbf{W}$ becomes diagonal. As $\mathbf{W}$ is a positive definite matrix and thence, it can be always diagonalized (as a real-valued matrix) with orthogonal eigenvectors. But the problem here is that the integer ambiguities $\mathbf{N}$ must be transformed preserving its integer nature!

Thence, we are looking for an "integer-valued" transformation matrix $\mathbf{Z}$ that makes the matrix $\mathbf{W}$ as close as possible to a diagonal matrix (decorrelating as much as possible the ambiguities) and with (as much as possible) similar axes (spherical).

$$\mathbf{N}' = \mathbf{Z}\,\mathbf{N} \qquad \hat{\mathbf{N}}' = \mathbf{Z}\,\hat{\mathbf{N}} \qquad \mathbf{P}_{\hat{\mathbf{N}}'} = \mathbf{Z}\,\mathbf{P}_{\hat{\mathbf{N}}}\,\mathbf{Z}^T$$

Moreover, the inverse of transformation matrix $\mathbf{Z}^{-1}$ must be also integer, to transform back the results after finding the ambiguities

Note that $\mathbf{Z},\ \mathbf{Z}^{-1}$ integers $\Rightarrow |\det(\mathbf{Z})| = 1$ (i.e. it is a volume-preserving transformation)



Pictures from [RD-6]

Exercise:

Show that:

$$\mathbf{Z},\ \mathbf{Z}^{-1}\ \text{Integers} \Rightarrow \left|\det(\mathbf{Z})\right| = 1$$

That is, $\mathbf{Z}$ is a volume-preserving transformation

From [RD-4]

Decorrelation: Computing the Z-transform

The following conditions must be fulfilled:

1. $\mathbf{Z}$ must have integer entries
2. $\mathbf{Z}$ must be invertible and have integer entries
3. The transformation $\mathbf{Z}$ must reduce the product of all ambiguity variances.

Note that $\mathbf{Z},\ \mathbf{Z}^{-1}$ integers $\Rightarrow |\det(\mathbf{Z})| = 1$
(i.e. it is a volume-preserving transformation)

Gauss manipulation over matrix $\mathbf{P}=\mathbf{W}^{-1}$ can be applied to find-out the matrix $\mathbf{Z}$.

$$\mathbf{P}_{\hat{\mathbf{N}}} = \begin{bmatrix} p_{\hat{N}_1\hat{N}_1} & p_{\hat{N}_1\hat{N}_2} \\ p_{\hat{N}_1\hat{N}_2} & p_{\hat{N}_2\hat{N}_2} \end{bmatrix}$$

$$\mathbf{Z}_1 = \begin{bmatrix} 1 & 0 \\ \alpha_1 & 1 \end{bmatrix}$$

➔ Transforms N_2 (N_1 remains unchanged)

$$\alpha_i = -\mathrm{int}\left[p_{\hat{N}_1\hat{N}_2} / p_{\hat{N}_i\hat{N}_i} \right]$$

$$\mathbf{Z}_2 = \begin{bmatrix} 1 & \alpha_2 \\ 0 & 1 \end{bmatrix}$$

➔ Transforms N_1 (N_2 remains unchanged)

Note: Inverse matrices have also integer entries

$$\mathbf{Z}_1^{-1} = \begin{bmatrix} 1 & 0 \\ -\alpha_1 & 1 \end{bmatrix} \qquad \mathbf{Z}_2^{-1} = \begin{bmatrix} 1 & -\alpha_2 \\ 0 & 1 \end{bmatrix}$$

Start transforming first the element with largest variance.

Gauss manipulation over matrix $\mathbf{P}=\mathbf{W}^{-1}$ can be applied to find-out the matrix $\mathbf{Z}$

$$\mathbf{P}_{\hat{\mathbf{N}}} = \begin{bmatrix} p_{\hat{N}_1\hat{N}_1} & p_{\hat{N}_1\hat{N}_2} \\ p_{\hat{N}_1\hat{N}_2} & p_{\hat{N}_2\hat{N}_2} \end{bmatrix}$$

$$\mathbf{Z}_1 = \begin{bmatrix} 1 & 0 \\ \alpha_1 & 1 \end{bmatrix}$$

➔ Transforms N_2 (N_1 remains unchanged)

$$\alpha_i = -\text{int}\left[p_{\hat{N}_1\hat{N}_2} / p_{\hat{N}_i\hat{N}_i} \right]$$

$$\mathbf{Z}_2 = \begin{bmatrix} 1 & \alpha_2 \\ 0 & 1 \end{bmatrix}$$

➔ Transforms N_1 (N_2 remains unchanged)

Example:

$$\hat{\mathbf{N}} = \begin{bmatrix} 1.05 \\ 1.30 \end{bmatrix} \qquad \mathbf{P}_{\hat{\mathbf{N}}} = \begin{bmatrix} 53.4 & 38.4 \\ 38.4 & 28.0 \end{bmatrix}$$

Example from [RD-4]

Step 1:

$$\mathbf{Z}_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \qquad \alpha_2 = -\text{int}\left[38.4/28.0\right] = -1$$

We transform first the element with largest variance (in this case N_1)

$$\mathbf{P}_{\hat{\mathbf{N}}'} = \mathbf{Z}_2\,\mathbf{P}_{\hat{\mathbf{N}}}\,\mathbf{Z}_2^T = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 53.4 & 38.4 \\ 38.4 & 28.0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4.6 & 10.4 \\ 10.4 & 28.0 \end{bmatrix}$$

The half, at most!

Step 2:

$$\mathbf{Z}_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \qquad \alpha_1 = -\text{int}\left[10.4/4.6\right] = -2$$

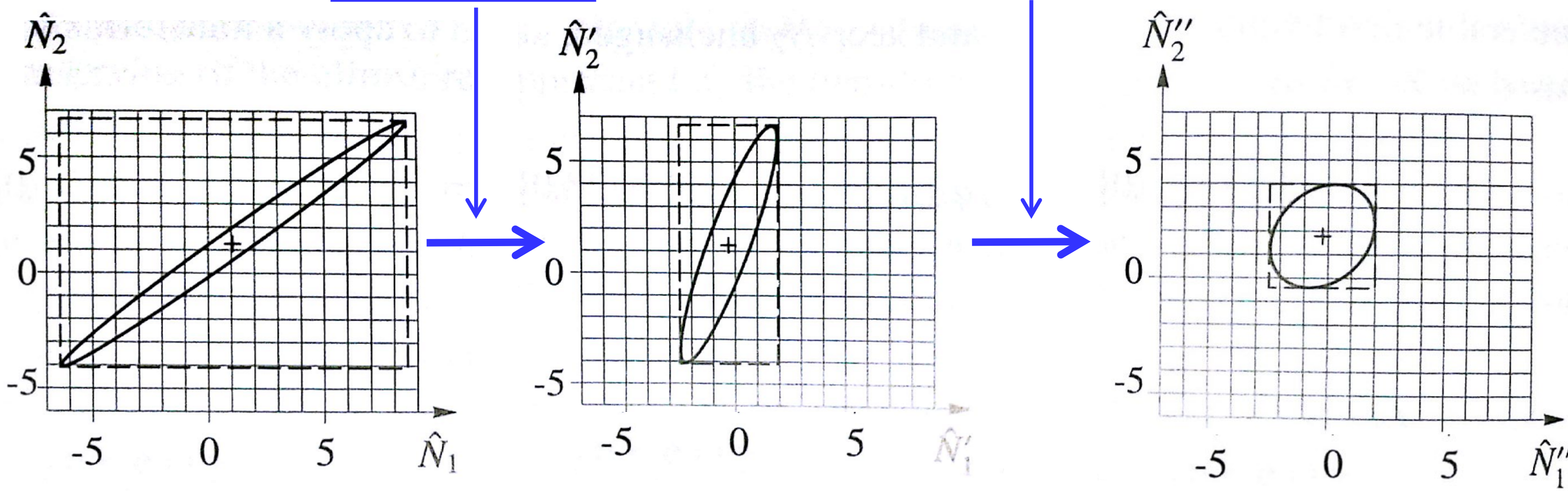
$$\mathbf{P}_{\hat{\mathbf{N}}''} = \mathbf{Z}_1\,\mathbf{P}_{\hat{\mathbf{N}}'}\,\mathbf{Z}_1^T = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 4.6 & 10.4 \\ 10.4 & 28.0 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4.6 & 1.2 \\ 1.2 & 4.8 \end{bmatrix}$$

In general, to increase the number of small off-diagonal elements, we have to transform first the elements with largest variance

gAGE/UPC research group of Astronomy and Geomatics
BarcelonaTECH, Spain

Example and pictures from [RD-4]

$$\mathbf{Z}_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{Z}_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$



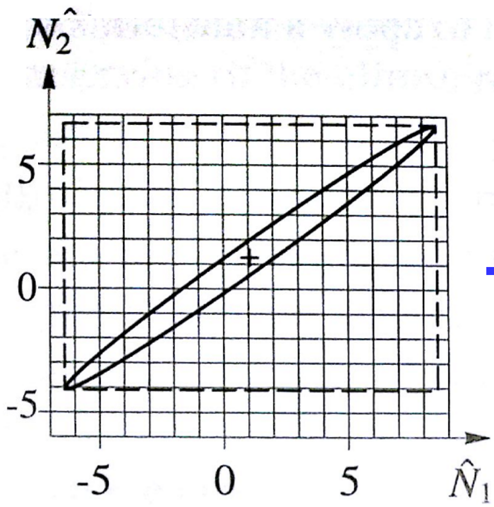
$$\mathbf{P}_{\hat{\mathbf{N}}} = \begin{bmatrix} 53.4 & 38.4 \\ 38.4 & 28.0 \end{bmatrix} \qquad \mathbf{P}_{\hat{\mathbf{N}}'} = \begin{bmatrix} 4.6 & 10.4 \\ 10.4 & 28.0 \end{bmatrix} \qquad \mathbf{P}_{\hat{\mathbf{N}}''} = \begin{bmatrix} 4.6 & 1.2 \\ 1.2 & 4.8 \end{bmatrix}$$

$$\mathbf{Z} = \mathbf{Z}_1\mathbf{Z}_2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

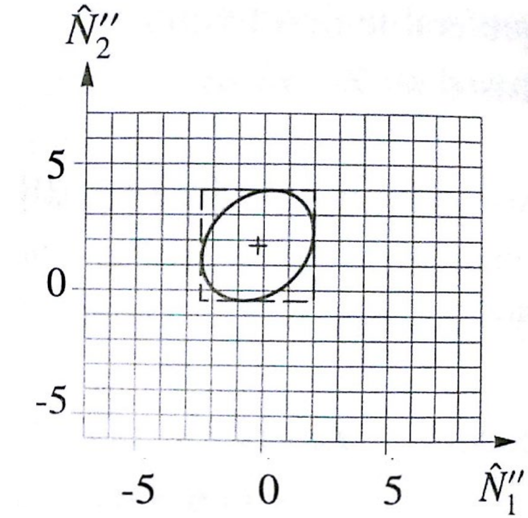
$$\mathbf{P}_{\hat{\mathbf{N}}''} = \mathbf{Z}\,\mathbf{P}_{\mathbf{N}}\,\mathbf{Z}^T = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 53.4 & 38.4 \\ 38.4 & 28.0 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 4.6 & 1.2 \\ 1.2 & 4.8 \end{bmatrix}$$

Example:

$$\hat{\mathbf{N}} = \begin{bmatrix} 1.05 \\ 1.30 \end{bmatrix} \qquad \mathbf{P}_{\hat{\mathbf{N}}} = \begin{bmatrix} 53.4 & 38.4 \\ 38.4 & 28.0 \end{bmatrix} \qquad \mathbf{P}_{\hat{\mathbf{N}}''} = \begin{bmatrix} 4.6 & 1.2 \\ 1.2 & 4.8 \end{bmatrix}$$



$$\mathbf{Z} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$



$$\hat{\mathbf{N}}'' = \mathbf{Z}\,\hat{\mathbf{N}} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1.05 \\ 1.30 \end{bmatrix} = \begin{bmatrix} -0.25 \\ 1.80 \end{bmatrix}$$

$$\mathbf{N}'' = \text{int} \begin{bmatrix} -0.25 \\ 1.80 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\mathbf{N} = \mathbf{Z}^{-1} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Let $\mathbf{P}$ be a symmetric and positive-definite matrix:

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

$\downarrow$

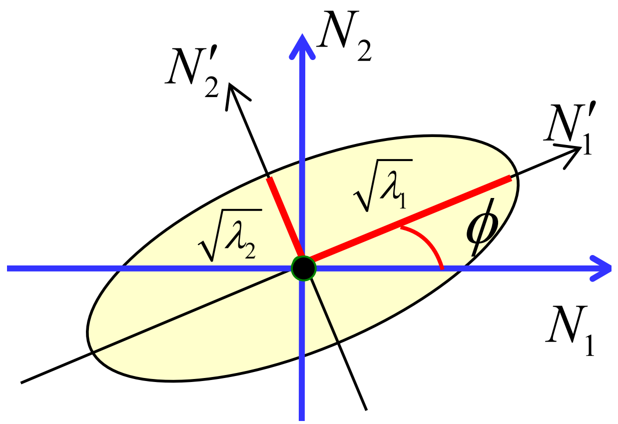
$$\mathbf{P}' = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$\lambda_1 = \frac{1}{2}\left(p_{11} + p_{22} + w\right)$$

$$\lambda_2 = \frac{1}{2}\left(p_{11} + p_{22} - w\right)$$

$$w = \sqrt{(p_{11} - p_{22})^2 + 4p_{12}^2}$$

$$\tan 2\phi = \frac{2p_{12}}{p_{11} - p_{22}}$$



Example:

$$\mathbf{P}_{\hat{\mathbf{N}}} = \begin{bmatrix} 53.4 & 38.4 \\ 38.4 & 28.0 \end{bmatrix} \qquad \hat{\mathbf{N}} = \begin{bmatrix} 1.05 \\ 1.30 \end{bmatrix}$$

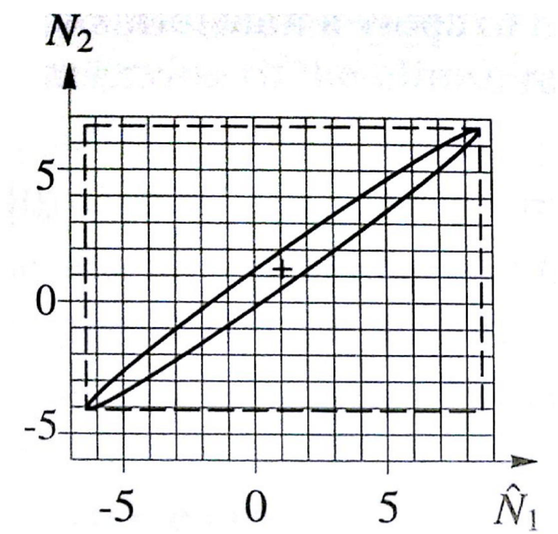
$\downarrow$

$$\mathbf{P}'_{\hat{\mathbf{N}}} = \begin{bmatrix} 81.14 & 0 \\ 0 & 0.25 \end{bmatrix}$$

$$\sqrt{\lambda_1} = \sqrt{81.14} = 9.0$$

$$\sqrt{\lambda_2} = \sqrt{0.25} = 0.5$$

$$\tan 2\phi = 3.02 \Rightarrow \phi = 35^{\circ}85$$



Consider again the previous problem of estimating $\Delta\mathbf{r}$, a *3*-vector of real numbers, and $\mathbf{N}$ a (*K-1)*-vector of integers, which are solution of

$$\mathbf{y} = \mathbf{G}\,\Delta\mathbf{r} + \lambda\,\mathbf{A}\,\mathbf{N} + \mathbf{v}$$

The solution comprises the following steps:

1. Obtain the float solution and its covariance matrix: $\begin{bmatrix} \Delta\hat{\mathbf{r}} \\ \hat{\mathbf{N}} \end{bmatrix} ; \begin{bmatrix} \mathbf{P}_{\Delta\hat{\mathbf{r}}} & \mathbf{P}_{\Delta\hat{\mathbf{r}},\hat{\mathbf{N}}} \\ \mathbf{P}_{\Delta\hat{\mathbf{r}},\hat{\mathbf{N}}} & \mathbf{P}_{\hat{\mathbf{N}}} \end{bmatrix}$

2. Find the integer vector $\mathbf{N}$ which minimizes the cost function

$$c(\mathbf{N}) = \left\|\mathbf{N}-\hat{\mathbf{N}}\right\|^2_{\mathbf{W}_{\hat{\mathbf{N}}}} = \left(\mathbf{N}-\hat{\mathbf{N}}\right)^T \mathbf{W}_{\hat{\mathbf{N}}} \left(\mathbf{N}-\hat{\mathbf{N}}\right) \qquad \mathbf{W}_{\hat{\mathbf{N}}} = \mathbf{P}^{-1}_{\hat{\mathbf{N}}}$$

   a) Decorrelation: Using the $\mathbf{Z}$ transform, the ambiguity search space is re-parametrized to decorrelate the float ambiguities.
   b) Integer ambiguities estimation (e.g. using sequential conditional least-squares adjustment, together with a discrete search strategy).
   c) Using the $\mathbf{Z}^{-1}$ transform, the ambiguities are transformed to the original ambiguity space.

3. Obtain the 'fixed' solution $\Delta\mathbf{r}$, from the fixed ambiguities $\mathbf{N}$.

$$\mathbf{y} - \lambda\,\mathbf{A}\,\mathbf{N} = \mathbf{G}\,\Delta\mathbf{r} + \mathbf{v}$$

gAGE/UPC research group of Astronomy and Geomatics
BarcelonaTECH, Spain

b) Integer ambiguities estimation

Several approach can be applied:

- Integer rounding
- Integer bootstrapping
- Integer Least-Squares
- …..

Comment:
In principle, the previous transformation $\mathbf{Z}$ *is not required by the estimation concept; it is only to achieve considerable gain in speed in the computation process [RD-5].*

b1) Integer rounding

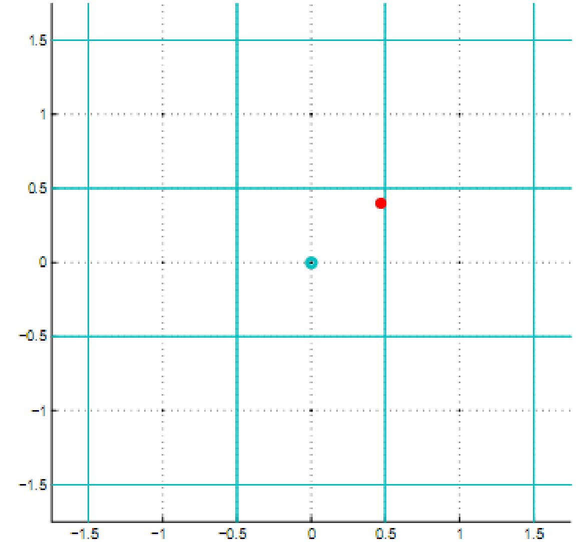
This is the simplest way.

Just to round-up the ambiguity vector entries to its nearest integer

$$\mathbf{N} = \left(\mathrm{int}(\hat{N}_1), \ldots, \mathrm{int}(\hat{N}_K)\right)$$

For instance, in the previous example:

$$\mathbf{N}'' = \mathrm{int}\begin{bmatrix} -0.25 \\ 1.80 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$



b2) Integer bootstrapping (from [RD-6])

It makes use of integer rounding, but it takes some of the correlations between the ambiguities into account.

1. We start with the most precise floated ambiguity (here we will assume $\hat{N}_n$)
2. Then, the remaining float ambiguities are corrected taking into account their correlation with the last ambiguity.

$$N_n = \text{int}\left[\hat{N}_n\right]$$

$$N_{n-1} = \text{int}\left[\hat{N}_{n-1|n}\right] = \text{int}\left[\hat{N}_{n-1} - \sigma_{\hat{N}_{n-1},\hat{N}_n}\sigma_{\hat{N}_n}^{-2}\left(\hat{N}_n - N_n\right)\right]$$

$$\vdots$$

$$N_1 = \text{int}\left[\hat{N}_{n|I}\right] = \text{int}\left[\hat{N}_1 - \sum_{i=2}^{n}\sigma_{\hat{N}_1,\hat{N}_{i|I}}\sigma_{\hat{N}_{i|I}}^{-2}\left(\hat{N}_{i|I} - N_i\right)\right]$$

Using the triangular decomposition

$$\mathbf{P}_{\hat{\mathbf{N}}} = \mathbf{L}^T\,\mathbf{D}\,\mathbf{L}$$

$$l_{ij} = \sigma_{\hat{N}_j,\hat{N}_{i|I}}\sigma_{\hat{N}_{i|I}}^{-2}$$

$\hat{N}_{i|I}$ Stands for the i-th ambiguity obtained through a conditioning of the previous $I=\{i+1,\ldots,n\}$ sequentially rounded ambiguities.

Example and pictures from [RD-6]

$$N_1 = \hat{N}_1 - \sigma_{\hat{N}_2,\hat{N}_1}\sigma^{-2}_{\hat{N}_2}\left(\hat{N}_2 - N_2\right)$$

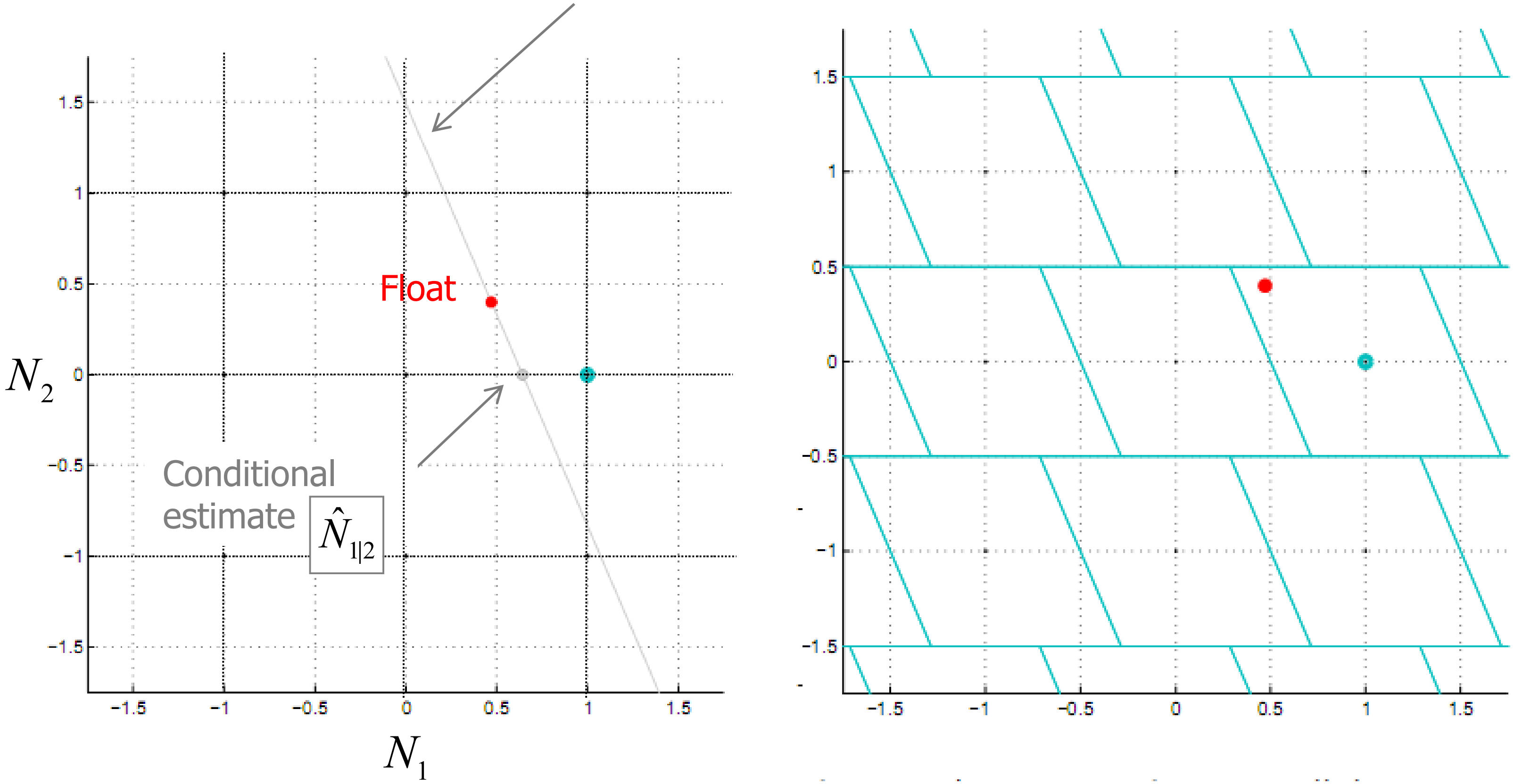


Figure 2.2: Principle and 2D pull-in regions for integer bootstrapping: parallelograms.

$$N_2 = \text{nint}\left[\hat{N}_2\right] = 0$$

$$N_1 = \text{nint}\left[\hat{N}_{1|2}\right] = \text{nint}\left[\hat{N}_1 - \sigma_{\hat{N}_2,\hat{N}_1}\sigma^{-2}_{\hat{N}_2}\left(\hat{N}_2 - N_2\right)\right] = 1$$

b3) Integer Least Squares (ISL) (from [RD-6])

1. The target to find the integer vector $\mathbf{N}$ which minimizes the cost function

$$c(\mathbf{N}) = \left\|\mathbf{N}-\hat{\mathbf{N}}\right\|^2_{\mathbf{P}^{-1}_{\hat{\mathbf{N}}}} = \left(\mathbf{N}-\hat{\mathbf{N}}\right)^T \mathbf{P}^{-1}_{\hat{\mathbf{N}}} \left(\mathbf{N}-\hat{\mathbf{N}}\right) \qquad \mathbf{W}_{\hat{\mathbf{N}}} = \mathbf{P}^{-1}_{\hat{\mathbf{N}}}$$

2. The integer minimiser is obtained through a search over the integer grid points on the n-dimensional hyper-ellipsoid: $\left(\mathbf{N}-\hat{\mathbf{N}}\right)^T \mathbf{P}^{-1}_{\hat{\mathbf{N}}} \left(\mathbf{N}-\hat{\mathbf{N}}\right) \leq \chi^2$

   - Where χ^2 determines the size of search region.
   - The solution is the integer grid point $\mathbf{N}$, inside the ellipsoid, giving the minimum value of cost function $c(\mathbf{N})$.

Using the triangular decomposition: $\mathbf{P}_{\hat{\mathbf{N}}} = \mathbf{L}^T\,\mathbf{D}\,\mathbf{L}$

where: $d_i = \sigma^{-2}_{\hat{N}_{i|I}}$

$l_{ij} = \sigma_{\hat{N}_j,\hat{N}_{i|I}}\,\sigma^{-2}_{\hat{N}_{i|I}}$

$$\left(\mathbf{N}-\hat{\mathbf{N}}\right)^T \mathbf{L}^{-1}\,\mathbf{D}^{-1}\,\mathbf{L}^{-T} \left(\mathbf{N}-\hat{\mathbf{N}}\right) \leq \chi^2 \rightarrow \left(\mathbf{N}-\bar{\mathbf{N}}\right)^T \mathbf{D}^{-1} \left(\mathbf{N}-\bar{\mathbf{N}}\right) \leq \chi^2$$

Defining:

$$\bar{\mathbf{N}} = \mathbf{N} - \mathbf{L}^{-T}(\mathbf{N}-\hat{\mathbf{N}}) \rightarrow \quad \mathbf{L}^T\left(\bar{\mathbf{N}}-\mathbf{N}\right) = (\hat{\mathbf{N}}-\mathbf{N})$$

$$c(N) = \frac{\left(N_1-\bar{N}_1\right)^2}{d_1} + \frac{\left(N_2-\bar{N}_2\right)^2}{d_2} + \cdots + \frac{\left(N_n-\bar{N}_n\right)^2}{d_n} \leq \chi^2$$

gAGE
gAGE/UPC research group of Astronomy and Geomatics
BarcelonaTECH, Spain

$$(\mathbf{N}-\bar{\mathbf{N}})^T \mathbf{D}^{-1}(\mathbf{N}-\bar{\mathbf{N}}) \leq \chi^2 \rightarrow c(N)=\frac{(N_1-\bar{N}_1)^2}{d_1}+\frac{(N_2-\bar{N}_2)^2}{d_2}+\cdots+\frac{(N_n-\bar{N}_n)^2}{d_n} \leq \chi^2$$

But $\tilde{N}_i$ depends on $\hat{N}_{i+1},\cdots,\hat{N}_n$.

$$\mathbf{L}^T(\bar{\mathbf{N}}-\mathbf{N})=(\hat{\mathbf{N}}-\mathbf{N}) \rightarrow \begin{cases}\bar{N}_n=\hat{N}_n \\ \bar{N}_i=\hat{N}_i+\sum_{j=i+1}^{n}\left(N_j-\hat{N}_j\right) l_{ji}; \quad i=n-1,n-2,\ldots,1\end{cases}$$

Search region bounds:

$$\bar{N}_n-d_n^{1/2}\chi \leq N_n \leq \bar{N}_n+d_n^{1/2}\chi$$

$$\bar{N}_{n-1}-d_{n-1}^{1/2}\left(\chi^2-\left(N_n-\hat{N}_n\right)^2 d_n\right)^{1/2} \leq N_{n-1} \leq \bar{N}_{n-1}+d_{n-1}^{1/2}\left(\chi^2-\left(N_n-\hat{N}_n\right)^2 d_n\right)^{1/2}$$

$$\vdots$$

$$\bar{N}_1-d_1^{1/2}\left(\chi^2-\sum_{j=2}^{n}\left(N_j-\hat{N}_j\right)^2 d_j\right)^{1/2} \leq N_1 \leq \bar{N}_n+d_1^{1/2}\left(\chi^2-\sum_{j=2}^{n}\left(N_j-\hat{N}_j\right)^2 d_j\right)^{1/2}$$

Acceptance test: The integer ambiguity solution corresponding to the smallest RMS residuals is used to select the candidate. However if two or more candidates give roughly similar values of RMS, the test can not be resolute.

➔A ratio test (of 2 or 3, depending of the algorithm) between the two smallest RMS is often used to validate the test.

Ellipsoid size: selecting the candidates for the acceptance test

The size of the ellipsoidal search region $\left(\mathbf{N}-\bar{\mathbf{N}}\right)^T \mathbf{D}^{-1}\left(\mathbf{N}-\bar{\mathbf{N}}\right) \le \chi^2$ is controlled by χ^2

Therefore, the performance of the search process is highly dependent on χ^2:

- A small χ^2 may result in a ellipsoidal region that fails to contain the solution.
- A too large value for χ^2 may result in high time-consuming for the search process.

Search with enumeration: When the number of required candidates is at most $n+1$ (with $n=dim(\mathbf{N})$), the following procedure can be applied to set the value χ^2:

- The best determined ambiguity is rounded to its nearest integer. The remaining ambiguities are then rounded using their correlations with the first ambiguity:

$$\bar{N}_n = \mathrm{nint}\left[\hat{N}_n\right]$$

$$\bar{N}_{n-1} = \mathrm{nint}\left[\hat{N}_{n-1|n}\right] = \mathrm{nint}\left[\hat{N}_{n-1} - \sigma_{\hat{N}_{n-1},\hat{N}_n}\sigma_{\hat{N}_n}^{-2}\left(\hat{N}_n - \bar{N}_n\right)\right]$$

$$\vdots$$

$$\bar{N}_1 = \mathrm{nint}\left[\hat{N}_{n|I}\right] = \mathrm{nint}\left[\hat{N}_1 - \sum_{i=2}^{n} \sigma_{\hat{N}_1,\hat{N}_{i|I}}\sigma_{\hat{N}_{i|I}}^{-2}\left(\hat{N}_{i|I} - \bar{N}_i\right)\right]$$

based on the bootstrapping estimator

- In each step of the conditional rounding procedure, two candidates are taken: The nearest and second-nearest, and conditional rounding is proceeded in both cases.

- If p candidates are requested, the values of cost function $c(\mathbf{N})$ are ordered in ascending order and χ^2 is chosen equal to the p-th value.

If more than $n+1$ candidates are requested, the volume of the search ellipsoid can be used ([RD-6]).

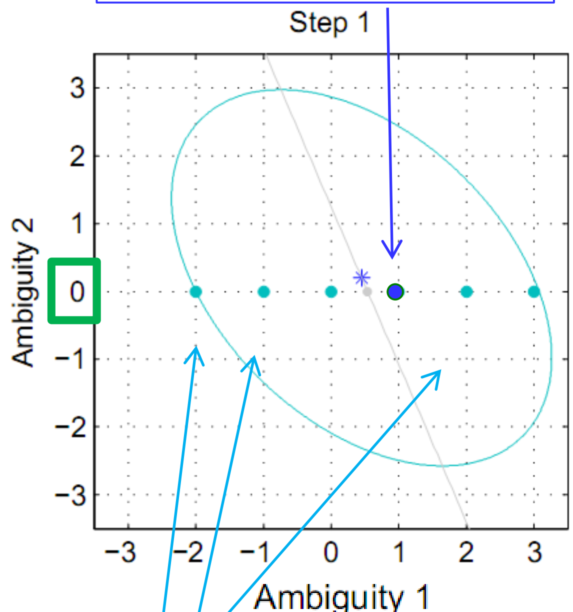
Search with shrinking technique: practical example

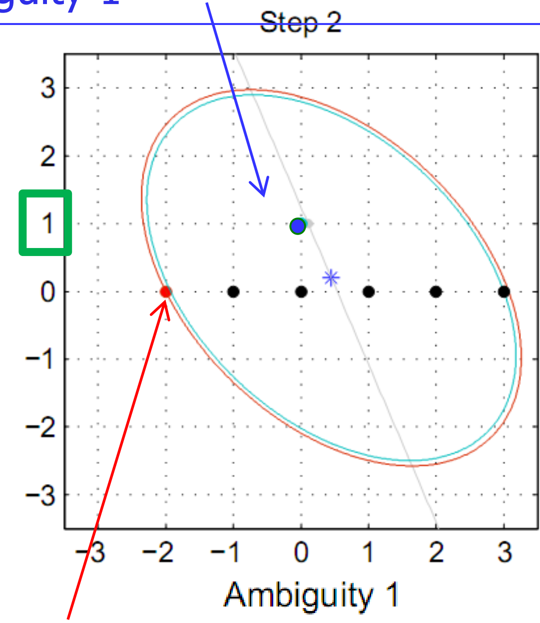
This is an alternative to the previous strategy, based on shrinking the search ellipsoid during the process of finding the candidates.

In the next example, we have to choose 6 candidates:

Fist candidate is the bootstrapped solution

Round "Ambiguity-2" to the second near integer and round the new conditional estimate for "Ambiguity-1"



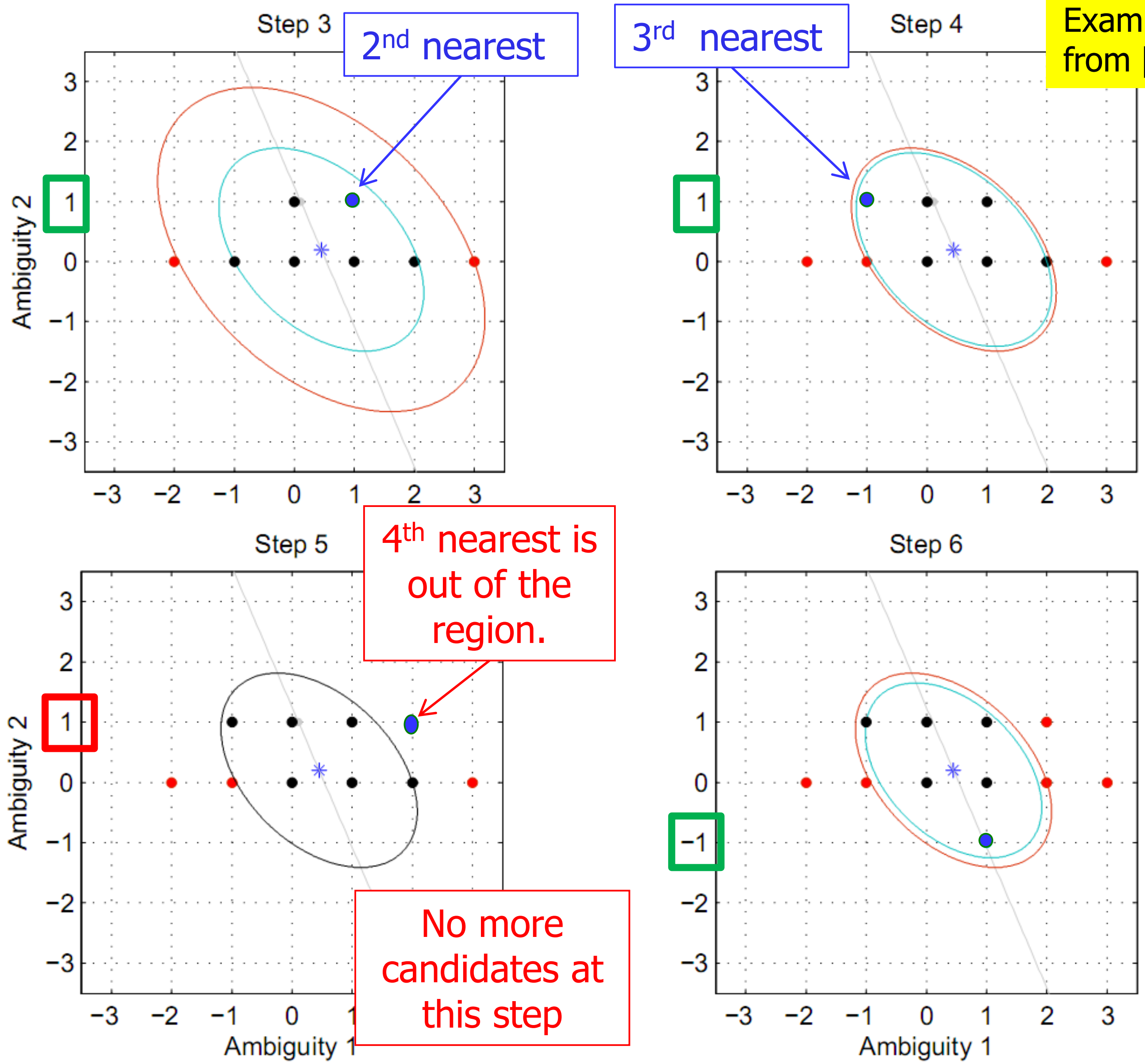


The other 5 candidates are found by choosing the conditional "Ambiguity-1" to the 2nd, 3rd, 4th and 5th nearest integers

The candidate with the largest χ^2 is removed.

Example and pictures from [RD-6]

Example and pictures from [RD-6]



Thence, the best 6 candidates are found (in the ISL sense). The one with the smallest cost function $c(\mathbf{N})$ value is the actual ISL solution.

Acceptance Test

The integer ambiguity solution corresponding to the smallest RMS residuals is used to select the candidate.

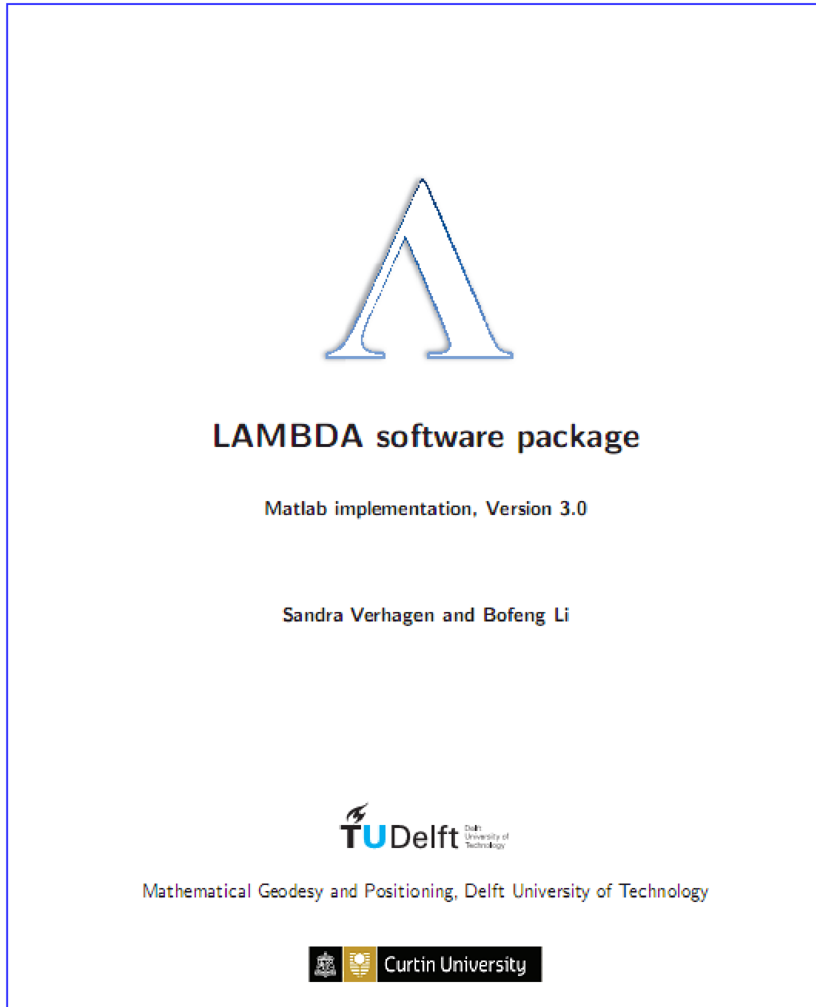
However if two or more candidates give roughly similar values of RMS, the test can not be resolutive.

→A ratio test (of 2 or 3, depending on the algorithm) between the two smallest RMS is often used to validate the test.

If the ratio is under these values, no integer solution can be determined and is better to use the floated solution.

$$RMS = \left\| \mathbf{N} - \hat{\mathbf{N}} \right\|_{\mathbf{P}_{\hat{\mathbf{N}}}^{-1}} = \sqrt{\left(\mathbf{N} - \hat{\mathbf{N}}\right)^T \mathbf{P}_{\hat{\mathbf{N}}}^{-1} \left(\mathbf{N} - \hat{\mathbf{N}}\right)}$$

Examples with MATLAB (octave)



LAMBDA software package

Matlab implementation, Version 3.0

Sandra Verhagen and Bofeng Li

TUDelft Delft University of Technology

Mathematical Geodesy and Positioning, Delft University of Technology

Curtin University

Note:
This document uses the transposed matrix $\mathbf{Z}^T$, but the principle is the same.

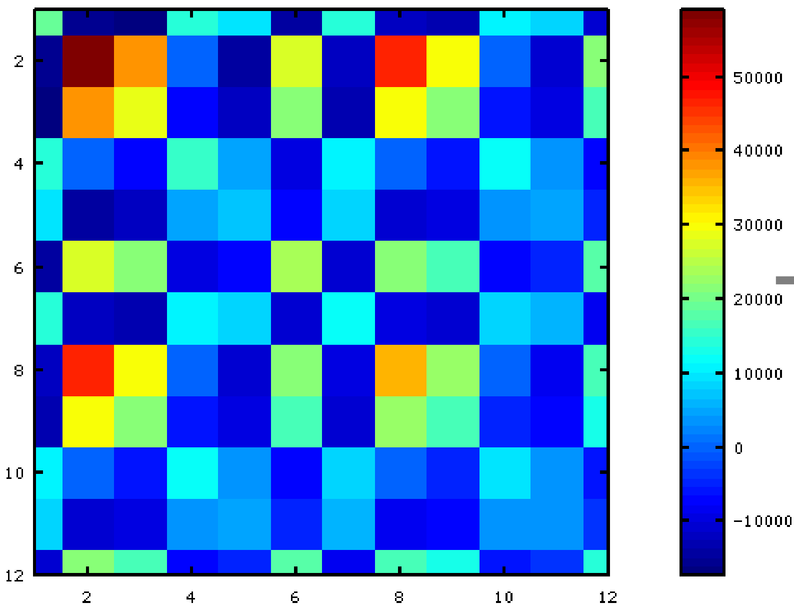
Examples with MATLAB (octave)

load large ➔ Q, a

$$Q \equiv \mathbf{P}_{\hat{\mathbf{N}}} = \mathbf{W}_{\hat{\mathbf{N}}}^{-1}$$

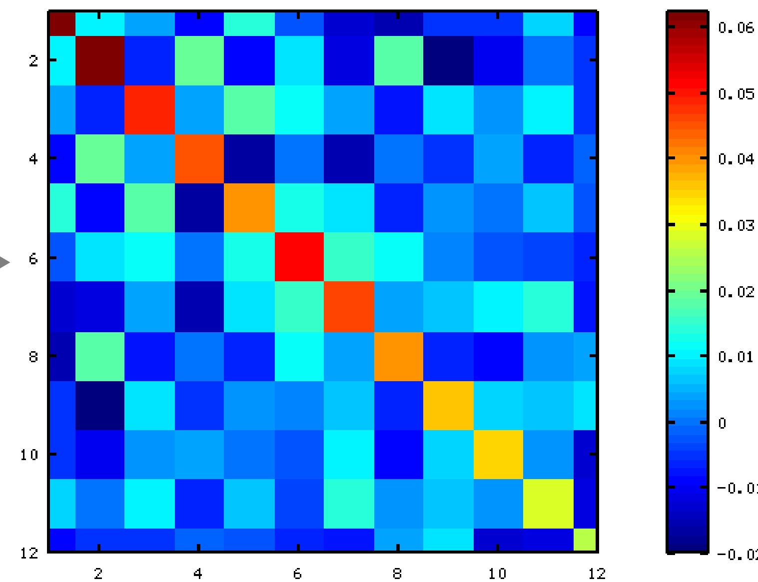
```
[Qz,Zt,Lz,Dz,az,iZ] = decorrel (Q,a);
```

```
imagesc(Q)
colorbar
```



Z →

```
imagesc(Qz)
colorbar
```




```
Qz=Lz'*diag(Dz)*Lz
```

gAGE/UPC research group of Astronomy and Geomatics
BarcelonaTECH, Spain

```
[Qz,Zt,Lz,Dz,az,iZ] = decorrel (Q,a);
```

```
Z=Zt'
```

```
Qz=Lz'*diag(Dz)*Lz
```

```
[L,D] = ldldecom(Q)

Q=L'*diag(D)*L
```

```
az= Z*a
a= inv(Z)*az
```

```
Qz= Z*Q*Z'
Q= inv(Z)*Qz* inv(Z')
```

```
Z =
     3     0    -4    -3    -5    -4    -4     2    -2     1    -3     1
    -0    -1     1    -1    -2     4     4    -3     4     1     0     1
     3     5    -2    -2     1    -1    -2     1    -1    -4    -1    -1
    -5    -2     3     2     4     3    -3    -2    -2     1    -3    -1
     4     5     1     4     2     6     5     2    -4     1     2    -4
    -8    -4     1     0     0    -3     2     3     2    -1    -0     4
     4    -7    -0     1     0    -4    -1    -7     3    -5    -1     2
     2    -1    -8    -1     2    -4     1     2    -4     2     2    -2
    -3     2     3    10    -8    -2    -5     0    -4     1    -4     0
    -1     6     8    -1     2     1     2     7     3    -2     6     1
    -8     7    -8     3    -6    -1     1     0     0     3    -1    -1
     8     1     6    -3     5     4    -5    -3     0    -0     1    -3
```

```
load large → Q, a
```

Integer rounding

```
round(a)
```

[- 28491 65753 38830 5004 -29196 -298 -22201 51236 30258 3899 -22749 -159]

Decorrelation + Integer rounding

```
[Qz,Zt,Lz,Dz,az,iZ] = decorrel (Q,a)
azfixed=round(az);
afixed=iZ*azfixed
```

[-28537 65473 38692 4939 -29228 -504 -22237 51018 30150 3849 -22774 -320]

Decorrelation + bootstrapping

```
[Qz,Zt,Lz,Dz,az,iZ] = decorrel (Q,a)
azfixed=bootstrap(az,Lz);
afixed=iZ*azfixed
```

[-28451 65749 38814 5025 -29165 -278 -22170 51233 30245 3916 -22725 -144]

Decorrelation + bootstrapping

```
[Qz,Zt,Lz,Dz,az,iZ] = decorrel (Q,a)
azfixed=bootstrap(az,Lz);
afixed=iZ*azfixed
```

[-28451 65749 38814 5025 -29165 -278 -22170 51233 30245 3916 -22725 -144]

Decorrelation + ILS with enumeration search

```
[Qz,Zt,Lz,Dz,az,iZ] = decorrel (Q,a);

[azfixed,sqnorm] = lsearch (az,Lz,Dz,6);
afixed=iZ*azfixed
```

													c(N)
-28451	65749	38814	5025	-29165	-278	-22170	51233	30245	3916	-22725	-144	→	**15.0**
-28279	65862	38805	5170	-29061	-192	-22036	51321	30238	4029	-22644	-77	→	31.6
-28727	65935	39032	4844	-29337	-178	-22385	51378	30415	3775	-22859	-66	→	33.9
-28546	66062	39027	4998	-29228	-83	-22244	51477	30411	3895	-22774	8	→	34.5
-28229	65518	38583	5197	-29056	-500	-21997	51053	30065	4050	-22640	-317	→	34.7
-28365	65586	38683	5084	-29124	-418	-22103	51106	30143	3962	-22693	-253	→	35.5

Decorrelation + bootstrapping

```
[Qz,Zt,Lz,Dz,az,iZ] = decorrel (Q,a)
azfixed=bootstrap(az,Lz);
afixed=iZ*azfixed
```

[-28451 65749 38814 5025 -29165 -278 -22170 51233 30245 3916 -22725 -144]

Decorrelation + ILS with search-and-shrink

```
[Qz,Zt,Lz,Dz,az,iZ] = decorrel (Q,a);

[azfixed,sqnorm] = ssearch (az,Lz,Dz,6);
afixed=iZ*azfixed
```

												c(N)
-28451	65749	38814	5025	-29165	-278	-22170	51233	30245	3916	-22725	-144	→ **15.0**
-28279	65862	38805	5170	-29061	-192	-22036	51321	30238	4029	-22644	-77	→ 31.6
-28727	65935	39032	4844	-29337	-178	-22385	51378	30415	3775	-22859	-66	→ 33.9
-28546	66062	39027	4998	-29228	-83	-22244	51477	30411	3895	-22774	8	→ 34.5
-28229	65518	38583	5197	-29056	-500	-21997	51053	30065	4050	-22640	-317	→ 34.7
-28365	65586	38683	5084	-29124	-418	-22103	51106	30143	3962	-22693	-253	→ 35.5

References

[RD-1] J. Sanz Subirana, J.M. Juan Zornoza, M. Hernández-Pajares, GNSS Data processing. Volume 1: Fundamentals and Algorithms. ESA TM-23/1. ESA Communications, 2013.

[RD-2] J. Sanz Subirana, J.M. Juan Zornoza, M. Hernández-Pajares, GNSS Data processing. Volume 2: Laboratory Exercises. ESA TM-23/2. ESA Communications, 2013.

[RD-3] Pratap Misra, Per Enge. Global Positioning System. Signals, Measurements, and Performance. Ganga-Jamuna Press, 2004.

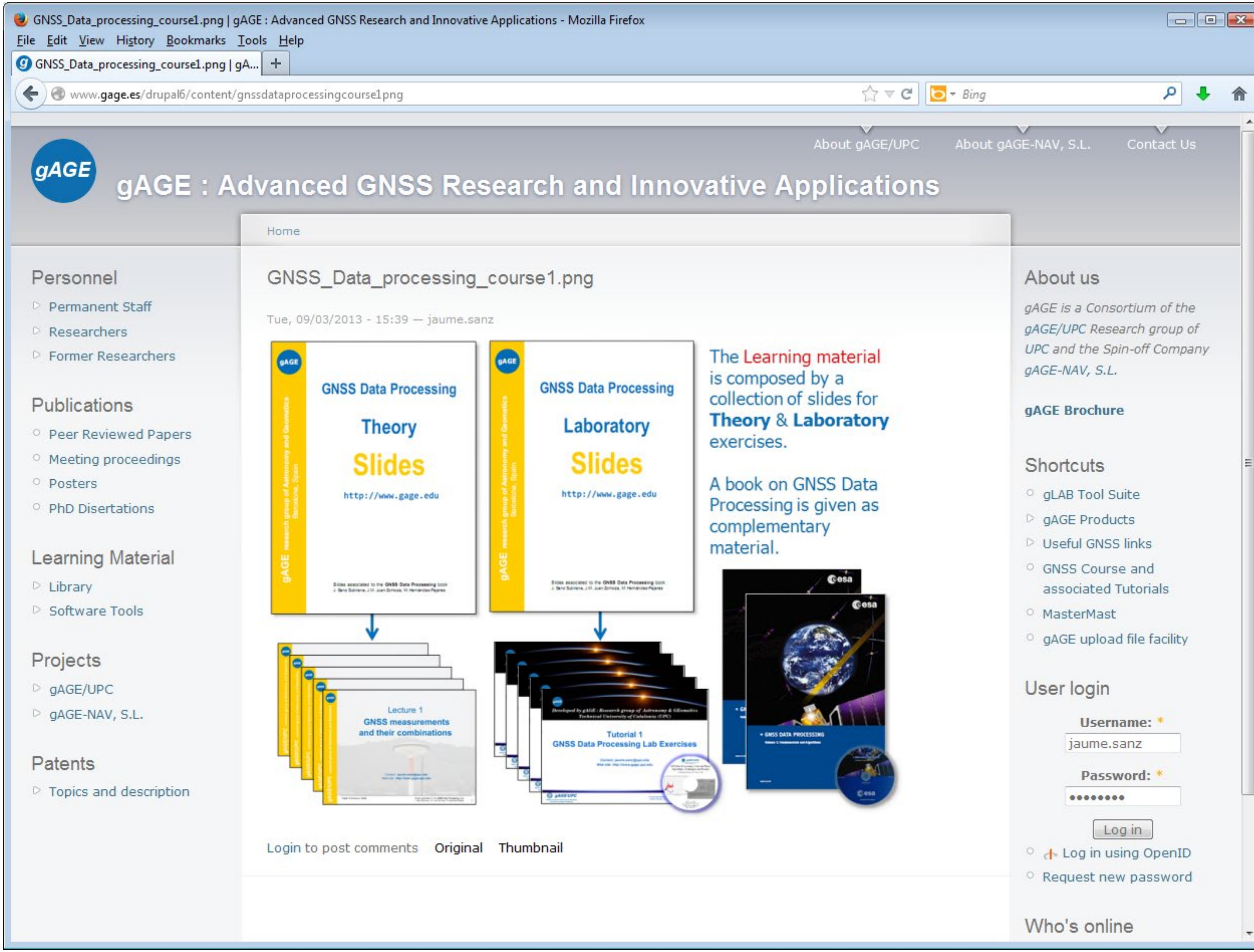
[RD-4] B. Hofmann-Wellenhof et al. GPS, Theory and Practice. Springer-Verlag. Wien, New York, 1994.

[RD-5] P. Joosten and Tiberius, C. LAMBDA: FAQs. GPS Solutions (2002), 6:109-114.

[RD-6] Sandra Verhagen and Bofeng L., LAMBDA software package. MATLAB implementation, Version 3.0. Mathematical Geodesy and Positioning, Delft University of Technology.

[RD-7] XW. Chang, CC. Paige, L. Yin, 2005, Code and carrier phase based short baseline GPS positioning: computational aspects. GPS Solutions, 2005 (2005) 9:72-83, DOI 10.1007/s10291-004-0112-8 Springer.

Thank you


GNSS_Data_processing_course1.png | gAGE : Advanced GNSS Research and Innovative Applications - Mozilla Firefox
File Edit View History Bookmarks Tools Help
www.gage.es/drupal6/content/gnssdataprocessingcourse1png
About gAGE/UPC
About gAGE-NAV, S.L.
Contact Us
gAGE : Advanced GNSS Research and Innovative Applications
Home
Personnel
Permanent Staff
Researchers
Former Researchers
Publications
Peer Reviewed Papers
Meeting proceedings
Posters
PhD Disertations
Learning Material
Library
Software Tools
Projects
gAGE/UPC
gAGE-NAV, S.L.
Patents
Topics and description
GNSS_Data_processing_course1.png
Tue, 09/03/2013 - 15:39 — jaume.sanz
GNSS Data Processing
Theory
Slides
http://www.gage.edu
GNSS Data Processing
Laboratory
Slides
http://www.gage.edu
The Learning material is composed by a collection of slides for Theory & Laboratory exercises.
A book on GNSS Data Processing is given as complementary material.
Lecture 1
GNSS measurements and their combinations
Tutorial 1
GNSS Data Processing Lab Exercises
Login to post comments
Original
Thumbnail
About us
gAGE is a Consortium of the gAGE/UPC Research group of UPC and the Spin-off Company gAGE-NAV, S.L.
gAGE Brochure
Shortcuts
gLAB Tool Suite
gAGE Products
Useful GNSS links
GNSS Course and associated Tutorials
MasterMast
gAGE upload file facility
User login
Username: *
jaume.sanz
Password: *
Log in
Log in using OpenID
Request new password
Who's online

GNSS Data Processing, Vol. 1: Fundamentals and Algorithms.
GNSS Data Processing, Vol. 2: Laboratory exercises.